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Criteria for the Finiteness of Restriction of $U(\mathfrak{g})$-Modules to Subalgebras and Applications to Harish-Chandra Modules.

A Study in Relation to the Associated Varieties

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Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Simple criteria are given for finitely generated $U(\mathfrak{g})$-modules H to remain finite under the restriction to subalgebras $A \subset U(\mathfrak{g})$, by using the algebraic varieties in $\mathfrak{g}^*$ associated to H and A. It is shown that, besides the finiteness, the $U(\mathfrak{g})$-modules H satisfying our criteria preserve the Gelfand–Kirillov dimension and Bernstein degree under the restriction to Lie subalgebras. Applying these results to Harish-Chandra modules of a semisimple Lie algebra $\mathfrak{g}$, we specify, among other things, a large class of Lie subalgebras of $\mathfrak{g}$ on which all the Harish-Chandra modules are of finite type. This allows us to extend the finite multiplicity theorems for induced representations of a semisimple Lie group, given in our earlier work [H. Yamashita, *J. Math. Kyoto Univ.* **28** (1988), 173–211; *J. Math. Kyoto Univ.* **28** (1988), 383–444].

INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The natural increasing filtration $(U_k(\mathfrak{g}))_{k=0,1,\dots}$ of $U(\mathfrak{g})$ defines a commutative graded ring $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_k U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$, which is isomorphic to the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ by the Poincaré–Birkhoff–Witt theorem. The identification $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ allows us to relate various objects in (non-commutative) enveloping algebra theory with those in commutative algebra and algebraic geometry for $S(\mathfrak{g})$ and the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ (see [2, 5, 15, 18, 19]).

For instance, if H is a $U(\mathfrak{g})$-module generated by a finite-dimensional subspace H_0, we can associate to the pair (H, H_0) a graded $S(\mathfrak{g})$-module of finite type by $\operatorname{gr}(H; H_0) := \bigoplus_k H_k/H_{k-1}$ with $H_k = U_k(\mathfrak{g})\, H_0$. The annihilator $J(H; H_0)$ of $\operatorname{gr}(H; H_0)$ in $S(\mathfrak{g})$ defines the *associated variety* $\mathscr{V}(\mathfrak{g}; H) \subset \mathfrak{g}^*$ of H, independent of H_0, as the set of common zeros of all

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elements of $J(H; H_0)$. The celebrated Hilbert–Serre theorem in commutative ring theory says that this variety $\mathscr{V}(\mathfrak{g}; H)$ supports well the graded $S(\mathfrak{g})$-module $\mathrm{gr}(H; H_0)$ (see Theorem 1.1).

In this paper, we give simple criteria for finitely generated $U(\mathfrak{g})$-modules H to remain finite under the restriction to subalgebras of $U(\mathfrak{g})$, by means of the associated varieties $\mathscr{V}(\mathfrak{g}; H)$. Applying the criteria, we specify, among other things, a large class of Lie subalgebras of a semisimple Lie algebra on which all the Harish-Chandra modules are of finite type. This extends some results of Casselman and Osborne [9] and Joseph [14] on the restriction of admissible modules to nilpotent Lie subalgebras appearing in the Iwasawa decomposition. Moreover we develop, with the help of Frobenius reciprocity, the finite multiplicity theorems for induced representations of a semisimple Lie group, obtained in our earlier work [21].

Let us now explain our basic ideas and the principal results of this article.

(A) For a subalgebra A of $U(\mathfrak{g})$ containing the identity element, let R denote the associated graded subalgebra $\mathrm{gr}\, A := \bigoplus_{k \geqslant 0} A_k/A_{k-1}$ of $S(\mathfrak{g})$ with $A_k = A \cap U_k(\mathfrak{g})$. We say that a finitely generated $U(\mathfrak{g})$-module H has the *good restriction* to A if there exists a finite-dimensional generating subspace H_0 of H for which the $S(\mathfrak{g})$-module $M := \mathrm{gr}\,(H; H_0)$ is of finite type over R. It is standard to verify that the original H is finitely generated over A if its restriction to A is good.

We can characterize the $U(\mathfrak{g})$-modules H having the good restriction to a given A, by using the associated varieties. To be specific, we first observe that the graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0)$ is finitely generated over R if and only if the quotient $S(\mathfrak{g})$-module $M/R_+ M$ is of finite-dimension over $\mathbf{C}$, where R_+ denotes the maximal graded ideal of R. Second, the Hilbert–Serre theorem (or Hilbert's Nullstellensatz) tells us that $\dim M/R_+ M < \infty$ whenever

$$\mathscr{V}(\mathfrak{g}; H) \cap R_+^{\#} = (0) \tag{0.1}$$

holds, where $R_+^{\#}$ denotes the algebraic variety of $\mathfrak{g}^*$ determined as the set of common zero points of elements of R_+. Furthermore, it is shown that the converse is also true provided that R is Noetherian. (See Proposition 2.1.)

Summing up the above discussion, we obtain the first main result of this paper, as follows.

THEOREM I (See Theorems 2.1 and 2.2(1)). (1) *A finitely generated $U(\mathfrak{g})$-module H has the good restriction to a subalgebra A whenever* (0.1) *is fulfilled for $R = \mathrm{gr}\, A$. The converse is also true if the ring R is Noetherian.*

(2) *The condition* (0.1) *guarantees that H is of finite type over A.*

If $A = U(\mathfrak{n})$ for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, then the corresponding graded ring $R = S(\mathfrak{n})$ is Noetherian and $R_+^\#$ equals the orthogonal $\mathfrak{n}^\perp$ of $\mathfrak{n}$ in $\mathfrak{g}^*$. Accordingly, one sees from Theorem I that H has the good restriction to $U(\mathfrak{n})$ if and only if $\mathscr{V}(\mathfrak{g}; H) \cap \mathfrak{n}^\perp = (0)$. In this case, we find that, besides the finiteness, H preserves some other invariants under the restriction to $U(\mathfrak{n})$:

THEOREM II (See Theorem 2.2(2)). *If the restriction of an H to $U(\mathfrak{n})$ is good, the Gelfand–Kirillov dimension $d(\mathfrak{n}; H) := \dim \mathscr{V}(\mathfrak{n}; H)$ and the Bernstein degree $c(\mathfrak{n}; H)$ (see* 1.2 *for the definition) of H as a $U(\mathfrak{n})$-module coincide, respectively, with those $d(\mathfrak{g}; H)$ and $c(\mathfrak{g}; H)$ as a $U(\mathfrak{g})$-module. Furthermore, the variety $\mathscr{V}(\mathfrak{g}; H)$ is carried into $\mathscr{V}(\mathfrak{n}; H)$ by the restriction of linear forms on $\mathfrak{g}$ to $\mathfrak{n}$.*

(B) The general results given in (A), have interesting applications to Harish-Chandra modules of a semisimple Lie algebra.

Now let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We denote by $\mathfrak{g}$ the complexified Lie algebra of $\mathfrak{g}_0$, and the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{h}(\subset \mathfrak{g})$, conventionally. By a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module is meant a finitely generated $U(\mathfrak{g})$-module H on which the subalgebra $U(\mathfrak{k})\,\mathscr{Z}(\mathfrak{g})$ acts locally finitely, where $\mathscr{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. We regard the variety $\mathscr{V}(\mathfrak{g}; H)$ as a subset of $\mathfrak{g}$ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$.

The following two facts are essential for our applications to Harish-Chandra modules.

(1) The associated variety $\mathscr{V}(\mathfrak{g}; H)$ of a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module H is contained in the variety $\mathscr{N}(\mathfrak{p})$ of all nilpotent elements in $\mathfrak{p}$ (Lemma 3.1).

(2) There exists a Harish-Chandra module $\tilde{H}$ for which $\mathscr{V}(\mathfrak{g}; \tilde{H})$ coincides with the whole $\mathscr{N}(\mathfrak{p})$ (Proposition 3.2).

These facts together with Theorem I yield the following

THEOREM III (See Theorem 3.1). *All the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction to a subalgebra A of $U(\mathfrak{g})$ if $\mathscr{N}(\mathfrak{p}) \cap R_+^\# = (0)$ holds for $R = \operatorname{gr} A$. The converse is also true when R is Noetherian.*

(C) Suggested by this theorem, we say that a Lie subalgebra $\mathfrak{n}_0$ is *large* in $\mathfrak{g}_0$ if there exists an inner automorphism x of $\mathfrak{g}_0$ such that

$$(x \cdot \mathfrak{n})^\perp \cap \mathscr{N}(\mathfrak{p}) = (0), \tag{0.2}$$

or equivalently, each Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module has the good restriction to $U(x \cdot \mathfrak{n})$.

We can specify many of large Lie subalgebras of $\mathfrak{g}_0$. At first, the maximal nilpotent Lie subalgebras and also the symmetrizing Lie subalgebras of $\mathfrak{g}_0$ are proved to be large in $\mathfrak{g}_0$ (Propositions 4.1 and 4.2). Theorems I and II applied to the former example cover results of Casselman and Osborne [9, Th. 2.3] and Joseph [14, II, 5.6]. Second, it is shown that the largeness of a Lie subalgebra is preserved by the parabolic induction (see 4.2).

Third, we say that a Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is *quasi-spherical* if there exists a minimal parabolic subalgebra $\mathfrak{q}_{m,0}$ of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{n}_0 + \mathfrak{q}_{m,0}$. Such Lie subalgebras give rise to the homogeneous spaces of a semisimple Lie group on which each minimal parabolic subgroup admits an open orbit (see, e.g., [3, 4, 6, 7, 16, 17]).

THEOREM IV (See Theorem 4.1). *Any quasi-spherical Lie subalgebra is large in* $\mathfrak{g}_0$.

One can see from Theorem III, coupled with a recent result of Bien and Oshima, that the converse is also true in the above theorem if a large Lie subalgebra $\mathfrak{n}_0$ is algebraic in $\mathfrak{g}_0$.

(D) Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We denote the corresponding Lie algebras by $\mathfrak{g}_0$ and $\mathfrak{k}_0$, respectively. By Harish-Chandra, the admissible Hilbert space G-representations correspond to Harish-Chandra $(\mathfrak{g}, K)$-modules, i.e., such $(\mathfrak{g}, \mathfrak{k})$-modules with compatible K-action, by passing to the K-finite part. On the other side, if (η, E) is a smooth Fréchet representation of a closed subgroup N of G, the space $\mathscr{A}(G; \eta)$ of real analytic sections of associated vector bundle $G \times_N E$ has a natural structure of compatible $(G, U(\mathfrak{g}))$-module (see 5.1).

With the aid of Frobenius reciprocity (cf. Proposition 5.1), Theorems I and III on the restriction of $U(\mathfrak{g})$-modules allow us to give finite multiplicity criteria for analytically induced modules $\mathscr{A}(G; \eta)$ (Theorems 5.3–5.5).

Among other things, we obtain the following

THEOREM V (See Theorem 5.5). *Let N be a closed subgroup of G whose Lie algebra $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, and take an $x \in G$ for which* $(\mathrm{Ad}(x)\,\mathfrak{n})^{\perp} \cap \mathscr{N}(\mathfrak{p}) = (0)$. *Then the intertwining number* $\dim \mathrm{Hom}_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$ *is finite for every Harish-Chandra $(\mathfrak{g}, K)$-module H, if the restriction of η to compact subgroup $N \cap x^{-1}Kx$ has the finite multiplicity property.*

This theorem extends one of the principal results in our previous work [21, I, Th. 2.12], where we studied the case of certain semidirect product large Lie subalgebras $\mathfrak{n}_0$, through the theory of (K, N)-spherical functions. Related to this result, we remark here that a sharp estimate of the intertwining number $\dim \mathrm{Hom}_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$ has been given recently by Bien and Oshima when $\mathfrak{n}_0$ is quasi-spherical and η is finite-dimensional, by quite a different method based on the boundary map in the Helgason conjecture.

The organization of this paper is as follows. We begin with preparing in Section 1 the notions and fundamental facts which we need throughout this article. Section 2 gives the theoretical basis of this work. We develop the general theory on restriction of $U(\mathfrak{g})$-modules to subalgebras by using the associated varieties. The criteria for good restriction to subalgebras are established in several forms in 2.1 and 2.2, and we clarify in 2.3 and 2.4 some properties of $U(\mathfrak{g})$-modules with good restriction.

In Section 3, applying the results of Section 2 to semisimple Lie algebras $\mathfrak{g}$, we characterize, in relation to the nilpotent variety $\mathcal{N}(\mathfrak{p})$, subalgebras of $U(\mathfrak{g})$ to which all the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction. The principal result of Section 3, Theorem 3.1, is presented in a much more general setting. Section 4 is devoted to the specification of large Lie subalgebras of a real semisimple Lie algebra. Section 5 develops finite multiplicity criteria for analytically induced representations of a (semisimple) Lie group, by making use of the results of Sections 2–4 and a reciprocity of Frobenius type.

The principal results of this article have been reported in the note [23].

1. Associated Varieties for Finitely Generated $U(\mathfrak{g})$-Modules

At first, we equip ourselves with some fundamental facts in commutative algebra and algebraic geometry, and introduce three important invariants: the associated variety, the Bernstein degree, and the Gelfand–Kirillov dimension, of finitely generated modules over a complex Lie algebra.

1.1. *The Hilbert–Serre Theorem*

Let V be a finite-dimensional complex vector space. We denote by $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ the symmetric algebra of V, where $S^k(V)$ is the subspace of $S(V)$ consisting of all homogeneous elements of degree k. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated, non-zero, graded $S(V)$-module, on which $S(V)$ acts in such a way as $S^k(V)\, M_{k'} \subset M_{k+k'}$ $(k, k' \geqslant 0)$. Then it is easy to see that each homogeneous component M_k is finite-dimensional. Set

$$\varphi_M(q) := \dim(M_0 + M_1 + \cdots + M_q) \tag{1.1}$$

for each integer $q \geqslant 0$.

Theorem 1.1 (Hilbert and Serre; see [24, Chap. VII, Sect. 12]). (1) *There exists a unique polynomial $\tilde{\varphi}_M(q)$ in q such that $\varphi_M(q) = \tilde{\varphi}_M(q)$ for sufficiently large q.*

(2) *Let* $(c(M)/d(M)!)\, q^{d(M)}$ *be the leading term of* $\tilde{\varphi}_M$. *Then* $c(M)$ *is a positive integer, and the degree* $d(M)$ *of this polynomial coincides with the dimension of the associated algebraic variety*

$$\mathscr{V}(M) := \{\lambda \in V^* \mid f(\lambda) = 0 \text{ for all } f \in \operatorname{Ann}_{S(V)} M\}. \tag{1.2}$$

Here $\operatorname{Ann}_{S(V)} M$ *denotes the annihilator of* M *in* $S(V)$, V^* *the dual space of* V, *and* $S(V)$ *is identified with the polynomial ring over* V^* *in the canonical way.*

Since the annihilator $\operatorname{Ann}_{S(V)} M$ is a graded ideal contained in $S(V)_+ := \bigoplus_{k>0} S^k(V)$, the variety $\mathscr{V}(M)$ is an algebraic cone in V^*. This combined with (2) of the above theorem gives in particular the following corollary, which is one of the keys for studying in Section 2 the restriction of $U(\mathfrak{g})$-modules to subalgebras.

COROLLARY 1.1. *A finitely generated, non-zero, graded* $S(V)$*-module* M *is finite-dimensional if and only if its associated variety* $\mathscr{V}(M)$ *equals* (0).

Remark. It is not difficult to deduce this corollary directly from *Hilbert's Nullstellensatz.*

1.2. *Associated Varieties for* $U(\mathfrak{g})$*-Modules*

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. For each integer $k \geqslant 0$, we denote by $U_k(\mathfrak{g})$ the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \cdots X_m$ with $m \leqslant k$ and $X_j \in \mathfrak{g}$ $(1 \leqslant j \leqslant m)$. One gets a natural increasing filtration $(U_k(\mathfrak{g}))_{k \geqslant 0}$ of $U(\mathfrak{g})$ such that

$$U(\mathfrak{g}) = \bigcup_{k=0}^{\infty} U_k(\mathfrak{g}), \qquad U_k(\mathfrak{g})\, U_m(\mathfrak{g}) = U_{k+m}(\mathfrak{g}),$$
$$[U_k(\mathfrak{g}), U_m(\mathfrak{g})] \subset U_{k+m-1}(\mathfrak{g}).$$

The associated graded commutative algebra $\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{k \geqslant 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$ $(U_{-1}(\mathfrak{g}) := (0))$ is isomorphic to the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k \geqslant 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$, by the Poincaré–Birkhoff–Witt theorem. We will identify these two algebras with each other.

Now let H be a finitely generated, non-zero $U(\mathfrak{g})$-module. Take a finite-dimensional generating subspace H_0 of H: $H = U(\mathfrak{g})\, H_0$. Setting $H_k = U_k(\mathfrak{g})\, H_0$ for $k = 1, 2, \ldots$, $H_{-1} = (0)$, one obtains an increasing filtration $(H_k)_k$ of H such that

$$H = \bigcup_{k=0}^{\infty} H_k, \qquad U_m(\mathfrak{g})\, H_k = H_{k+m}. \tag{1.3}$$

Correspondingly, we have a graded $S(\mathfrak{g})$-module

$$M = \bigoplus_k M_k \qquad \text{with} \quad M_k = H_k/H_{k-1}, \tag{1.4}$$

which will be denoted by $\operatorname{gr}(H; H_0)$ because the above filtration of H is determined by H_0. Since $M_k = S^k(\mathfrak{g})\, M_0$, M is finitely generated over $S(\mathfrak{g})$. So we can define for this M the variety $\mathscr{V}(M) \subset \mathfrak{g}^*$, the integers $c(M)$ and $d(M)$ as in Theorem 1.1. It is easy to see that these quantities are independent of the choice of a generating subspace H_0. Hereafter, we will denote these three invariants of H respectively by $\mathscr{V}(\mathfrak{g}; H)$, $c(\mathfrak{g}; H)$, and by $d(\mathfrak{g}; H)$, emphasizing that H is being considered as a $U(\mathfrak{g})$-module.

DEFINITION (cf. [5, III; 18, 19]). For a finitely generated non-zero $U(\mathfrak{g})$-module H, $\mathscr{V}(\mathfrak{g}; H)$, $c(\mathfrak{g}; H)$, and $d(\mathfrak{g}; H)$ $(= \dim \mathscr{V}(\mathfrak{g}; H)$ by Theorem 1.1(2)) are called respectively the *associated variety*, the *Bernstein degree*, and the *Gelfand–Kirillov dimension* of H.

Remark. It follows from Corollary 1.2 that H is finite-dimensional if and only if $\mathscr{V}(\mathfrak{g}; H) = (0)$, or equivalently its Gelfand–Kirillov dimension $d(\mathfrak{g}; H)$ equals 0.

2. RESTRICTION OF $U(\mathfrak{g})$-MODULES TO SUBALGEBRAS

Let A be a subalgebra of $U(\mathfrak{g})$ containing the identity element $1 \in U(\mathfrak{g})$. Denote by $\operatorname{gr} A = \bigoplus_{k \geqslant 0} A_k/A_{k-1}$ with $A_k = A \cap U_k(\mathfrak{g})$, the graded subalgebra of $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ associated to A. We say that a finitely generated $U(\mathfrak{g})$-module H has the *good restriction* to A if there exists a finite-dimensional generating subspace H_0 of H for which the associated graded $S(\mathfrak{g})$-module $\operatorname{gr}(H; H_0)$ is finitely generated over $\operatorname{gr} A$.

This section characterizes, by means of the associated varieties, $U(\mathfrak{g})$-modules H having the good restriction to A (Theorem 2.1). We show that such H's are finitely generated over A (Theorem 2.2(1)). Some more properties of these modules H are specified in 2.3–2.4.

2.1. *Restriction of $S(V)$-Modules to Graded Subalgebras*

We first discuss the restriction of graded $S(V)$-modules, where V is any complex vector space of finite dimension. Let $R = \bigoplus_{k \geqslant 0} R_k$, $R_k \subset S^k(V)$, be a graded subalgebra of $S(V)$ containing the identity element $1 \in S(V)$. $R_+ = \bigoplus_{k > 0} R_k$ denotes the maximal homogeneous ideal of R without constant term. We set for any subset Q of $S(V)$,

$$Q^\# := \{\lambda \in V^* \mid f(\lambda) = 0 \text{ for all } f \in Q\}. \tag{2.1}$$

Let M be, as in 1.1, a finitely generated, non-zero, graded $S(V)$-module. We consider the following four conditions on M in relation to R:

(a) $\mathscr{V}(M) \cap R_+^\# = (0)$, where $R_+^\# := (R_+)^\#$, and $\mathscr{V}(M) = (\mathrm{Ann}_{S(V)} M)^\#$ is the associated variety of M defined in (1.2).

(b) The ideal $\mathrm{Ann}_{S(V)} M + R_+ S(V)$ is of finite codimension in $S(V)$.

(c) The $S(V)$-submodule $R_+ M$ is of finite codimension in M.

(d) M is finitely generated as an R-module.

Then we get the following proposition on the relation among these conditions.

PROPOSITION 2.1. (1) *The condition* (a) (*resp.* (c)) *is equivalent to* (b) (*resp.* (d)). *Moreover,* (a) ($\Leftrightarrow$ (b)) *implies* (c) ($\Leftrightarrow$ (d)).

(2) *If the ring R is Noetherian, four conditions* (a)–(d) *are equivalent to each other.*

Proof. (1) Note that the associated variety of graded $S(V)$-module $S(V)/(\mathrm{Ann}_{S(V)} M + R_+ S(V))$ is equal to $\mathscr{V}(M) \cap R_+^\#$. By Corollary 1.1, this $S(V)$-module is finite-dimensional if and only if $\mathscr{V}(M) \cap R_+^\# = (0)$. Hence conditions (a) and (b) are equivalent.

We can show (a) $\Rightarrow$ (c) just in the same way, since the associated variety of $M/R_+ M$ is clearly contained in $\mathscr{V}(M) \cap R_+^\#$.

To show the equivalence (c) $\Leftrightarrow$ (d), assume that (c) holds. Then there exists a finite-dimensional graded subspace $Y = \bigoplus_k Y_k$, $Y_k \subset M_k$, of M such that $M = Y + R_+ M$. Taking the homogeneous components, we get

$$M_k = Y_k + \sum_{j=0}^{k-1} R_{k-j} M_j \qquad \text{for each } k. \tag{2.2}$$

By using this decomposition repeatedly, one deduces

$$M_k = \sum_{j=0}^{k} R_{k-j} Y_j, \tag{2.3}$$

and so $M = RY$. We thus obtain (d). The implication (d) $\Rightarrow$ (c) is immediate, and hence the condition (c) is equivalent to (d).

(2) Assume that R is a Noetherian ring and that M is of finite type over R. Let $(v_1, v_2, ..., v_n)$ be a finite system of homogeneous elements of M which generates M over $S(V)$: $M = S(V)\, v_1 + \cdots + S(V)\, v_n$. It follows from the above two assumptions that each submodule $S(V)\, v_i$ of M is finitely generated over R. By noting that $S(V)\, v_i \cong S(V)/\mathrm{Ann}_{S(V)}(v_i)$ as $S(V)$-modules, we see easily that

$$\dim(S(V))/(\mathrm{Ann}_{S(V)}(v_i) + R_+ S(V)) < \infty, \tag{2.4}$$

which is, again by Corollary 1.1, equivalent to $\mathscr{V}(S(V)\,v_i) \cap R_+^{\#} = (0)$. This implies (a), since $\mathscr{V}(M) = \bigcup_i \mathscr{V}(S(V)\,v_i)$. Thus we have seen (d) $\Rightarrow$ (a). In view of (1) proved above, we complete the proof. Q.E.D.

COROLLARY 2.1. *For a vector subspace W of V, set $W^{\perp} = \{\lambda \in V^* \mid \langle \lambda, w \rangle = 0$ for all $w \in W\}$. A finitely generated graded $S(V)$-module M, $\neq (0)$, is of finite type over the subalgebra $S(W)$ if and only if $\mathscr{V}(M) \cap W^{\perp} = (0)$.*

Proof. This corollary follows immediately from Proposition 2.1(2) by noting that $S(W)$ is Noetherian and that $W^{\perp} = S(W)_+^{\#}$. Q.E.D.

2.2. *Good Restriction of $U(\mathfrak{g})$-Modules*

Let $\mathfrak{g}$ be, as in 1.2, any complex Lie algebra, and let H be a finitely generated, non-zero $U(\mathfrak{g})$-module. Proposition 2.1 gives the following criterion for H to have the good restriction to a subalgebra of $U(\mathfrak{g})$.

THEOREM 2.1. *Let A be a subalgebra of $U(\mathfrak{g})$ containing the identity element $1 \in U(\mathfrak{g})$.*

(1) *The restriction of H to A is good whenever the condition*

$$\mathscr{V}(\mathfrak{g}; H) \cap R_+^{\#} = (0) \tag{2.5}$$

on algebraic varieties in $\mathfrak{g}^*$ is satisfied. Here $\mathscr{V}(\mathfrak{g}; H)$ is the associated variety of H defined in 1.2, and $R = \operatorname{gr} A$ denotes the graded subalgebra of $S(\mathfrak{g})$ associated to A.

(2) *Conversely, if R is Noetherian and if H admits the good restriction to A, one necessarily has (2.5).*

Proof. (1) Let H_0 be a finite-dimensional generating subspace of the $U(\mathfrak{g})$-module H, and let $M = \operatorname{gr}(H; H_0)$ be the associated graded $S(\mathfrak{g})$-module defined in (1.4). By definition one has $\mathscr{V}(\mathfrak{g}; H) = \mathscr{V}(M)$. In view of Proposition 2.1(1)[(a) $\Rightarrow$ (d)], we see that condition (2.5) assures that M is finitely generated over R. Hence the restriction of H to A is good.

(2) If H has the good restriction to A, there exists, by definition, an H_0 for which $M = \operatorname{gr}(H; H_0)$ is of finite type as an R-module. This together with Proposition 2.1(2)[(d) $\Rightarrow$ (a)] implies (2.5) under the assumption that R is Noetherian. Q.E.D.

Remark. As shown in the above proof, condition (2.5) guarantees that the graded $S(\mathfrak{g})$-module $\operatorname{gr}(H; H_0)$ is finitely generated over $R = \operatorname{gr} A$ for *every* generating subspace H_0 of H.

Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$. Applying Theorem 2.1 to the case $A = U(\mathfrak{n})$ ($R = S(\mathfrak{n})$ is obviously Noetherian), we obtain immediately the following

COROLLARY 2.2. *A finitely generated $U(\mathfrak{g})$-module H, $\neq (0)$, has the good restriction to $U(\mathfrak{n})$ if and only if $\mathscr{V}(\mathfrak{g}; H) \cap \mathfrak{n}^{\perp} = (0)$ holds.*

For later applications in Section 3, we give here another consequence of Theorem 2.1. Let B, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$, and let $C(B)$ denote the category of finitely generated $U(\mathfrak{g})$-modules H on which B acts locally finitely:

$$\dim Bv < \infty \qquad \text{for all} \quad v \in H.$$

We can (and do) take, for such an H, a finite-dimensional B-stable generating subspace $H_0 \subset H$. Set $Q = \operatorname{gr} B$. Then it is easily verified that the corresponding graded $S(\mathfrak{g})$-module $M = \operatorname{gr}(H; H_0)$ is annihilated by the maximal graded ideal Q_+ of Q. Hence one gets

$$\mathscr{V}(\mathfrak{g}; H) \subset Q_+^{\#}. \tag{2.6}$$

DEFINITION. We say that a subalgebra A of $U(\mathfrak{g})$ is *large* relative to B if all the $U(\mathfrak{g})$-modules H in the category $C(B)$ have the good restriction to A.

From (2.6) combined with Theorem 2.1, we deduce

PROPOSITION 2.2. *Let B, $Q = \operatorname{gr} B$ be as above, and let A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$ for which $R = \operatorname{gr} A$ is Noetherian. Then A is large relative to B if and only if*

$$\mathscr{V}_B \cap R_+^{\#} = (0) \tag{2.7}$$

holds for the subset $\mathscr{V}_B := \bigcup_H \mathscr{V}(\mathfrak{g}; H)$ of $Q_+^{\#}$, where H runs over the $U(\mathfrak{g})$-modules in $C(B)$. In particular, this is the case if $Q_+^{\#} \cap R_+^{\#} = (0)$.

Remark. It can be interesting to describe the subvariety $\mathscr{V}_B$ of $Q_+^{\#}$. We will show that $\mathscr{V}_B = Q_+^{\#}$ holds for the category $C(B)$ of Harish-Chandra modules of a semisimple Lie algebra $\mathfrak{g}$ (see Corollary 3.1).

Now define the double regular representation of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $\mathscr{U} := U(\mathfrak{g})$ by

$$(D_1 \otimes D_2)v = D_1 v\,{}'D_2 \qquad \text{for} \quad D_1, D_2 \in U(\mathfrak{g}) \text{ and } v \in \mathscr{U}.$$

Here $D \to {}'D$ denotes the principal anti-automorphism of $U(\mathfrak{g})$, characterized by ${}'X = -X$ for $X \in \mathfrak{g}$. Identifying $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ with $U(\mathfrak{g} \oplus \mathfrak{g})$ by the Poincaré–Birkhoff–Witt theorem, we regard $\mathscr{U}$ as a $U(\mathfrak{g} \oplus \mathfrak{g})$-module generated by the identity element $1 \in \mathscr{U}$.

The condition $Q_+^{\#} \cap R_+^{\#} = (0)$ in Proposition 2.2 can be related with the good restriction property of this module $\mathscr{U}$, as follows.

PROPOSITION 2.3. *Let A, B ($\ni 1$) be two subalgebras of $U(\mathfrak{g})$. The restriction of $U(\mathfrak{g}\oplus\mathfrak{g})$-module $\mathscr{U}$ to the subalgebra $A\otimes B$ is good if $Q_+^\# \cap R_+^\# = (0)$ is satisfied, where $R = \operatorname{gr} A$ and $Q = \operatorname{gr} B$. The converse is also true if $R\otimes Q$ is Noetherian.*

For the proof we prepare

LEMMA 2.1. *The associated variety $\mathscr{V}(\mathfrak{g}\oplus\mathfrak{g};\mathscr{U})$ of $\mathscr{U}$ is equal to* $\operatorname{adiag}(\mathfrak{g}^*\oplus\mathfrak{g}^*) := \{(\lambda, -\lambda) \mid \lambda\in\mathfrak{g}^*\}$.

Proof. Let J denote the annihilator of the element $1\in\mathscr{U}$ in $U(\mathfrak{g}\oplus\mathfrak{g})$. Then one has

$$\mathscr{U} = U(\mathfrak{g}\oplus\mathfrak{g})\cdot 1 \simeq U(\mathfrak{g}\oplus\mathfrak{g})/J$$

as $U(\mathfrak{g}\oplus\mathfrak{g})$-modules. It is immediate to verify that

$$J\supset U(\mathfrak{g}\oplus\mathfrak{g})\operatorname{diag}(\mathfrak{g}\oplus\mathfrak{g}), \tag{2.8}$$

where $\operatorname{diag}(\mathfrak{g}\oplus\mathfrak{g}) := \{(X, X) \mid X\in\mathfrak{g}\}$.

Let $\operatorname{gr}\mathscr{U} = S(\mathfrak{g}\oplus\mathfrak{g})/(\operatorname{gr} J)$ be the graded $S(\mathfrak{g}\oplus\mathfrak{g})$-module associated to $\mathscr{U}$ through the natural filtration $(U_k(\mathfrak{g}\oplus\mathfrak{g}))_{k\geqslant 0}$ of $U(\mathfrak{g}\oplus\mathfrak{g})$. It follows from (2.8) that $\operatorname{gr} J\supset S(\mathfrak{g}\oplus\mathfrak{g})\operatorname{diag}(\mathfrak{g}\oplus\mathfrak{g})$. This implies that

$$\mathscr{V}(\mathfrak{g}\oplus\mathfrak{g};\mathscr{U})\subset\operatorname{adiag}(\mathfrak{g}^*\oplus\mathfrak{g}^*), \tag{2.9}$$

since $\operatorname{adiag}(\mathfrak{g}^*\oplus\mathfrak{g}^*)$ is the orthogonal of $\operatorname{diag}(\mathfrak{g}\oplus\mathfrak{g})$ in $\mathfrak{g}^*\oplus\mathfrak{g}^*\simeq(\mathfrak{g}\oplus\mathfrak{g})^*$.

Noting that $U_k(\mathfrak{g}\oplus\mathfrak{g})\cdot 1 = U_k(\mathfrak{g})$ for every integer $k\geqslant 0$, one finds that the Gelfand–Kirillov dimension of our $U(\mathfrak{g}\oplus\mathfrak{g})$-module $\mathscr{U}$ is just $d := \dim\mathfrak{g}$. It follows from Theorem 1.1(2) that

$$\dim\mathscr{V}(\mathfrak{g}\oplus\mathfrak{g};\mathscr{U}) = d = \dim\operatorname{adiag}(\mathfrak{g}^*\oplus\mathfrak{g}^*).$$

We thus conclude that the two varieties in (2.9) coincide with one another, for $\operatorname{adiag}(\mathfrak{g}^*\oplus\mathfrak{g}^*)$ is an irreducible algebraic variety of dimension d.

Q.E.D.

Proof of Proposition 2.3. Observe that the graded subalgebra of $S(\mathfrak{g}\oplus\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g}\oplus\mathfrak{g})$ associated to $A\otimes B$ is given as

$$\operatorname{gr}(A\otimes B) = (\operatorname{gr} A)\otimes(\operatorname{gr} B) = R\otimes Q.$$

This coupled with Lemma 2.1 implies that

$$\mathscr{V}(\mathfrak{g}\oplus\mathfrak{g};\mathscr{U})\cap(\operatorname{gr}(A\otimes B))_+^\# = \{(\lambda, -\lambda) \mid \lambda\in R_+^\#\cap Q_+^\#\}.$$

We get the claims from Theorem 2.1. Q.E.D.

2.3. *Properties of $U(\mathfrak{g})$-Modules with Good Restriction*

The $U(\mathfrak{g})$-modules admitting the good restriction enjoy some nice properties as follows.

THEOREM 2.2. *Let H be a finitely generated, non-zero $U(\mathfrak{g})$-module having the good restriction to a subalgebra $A \subset U(\mathfrak{g})$. Then,*

(1) *H is finitely generated as an A-module.*

(2) *Assume that $A = U(\mathfrak{n})$ for some Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ (see Corollary 2.2). By (1), H is of finite type over $U(\mathfrak{n})$, and so one can define the associated variety $\mathscr{V}(\mathfrak{n}; H)$, Bernstein degree $c(\mathfrak{n}; H)$, and Gelfand–Kirillov dimension $d(\mathfrak{n}; H)$ of H as a $U(\mathfrak{n})$-module as well as such invariants as a $U(\mathfrak{g})$-module. These two kinds of invariants have the relations*

$$c(\mathfrak{g}; H) = c(\mathfrak{n}; H), \qquad d(\mathfrak{g}; H) = d(\mathfrak{n}; H), \tag{2.10}$$

and hence

$$\dim \mathscr{V}(\mathfrak{g}; H) = \dim \mathscr{V}(\mathfrak{n}; H). \tag{2.11}$$

Moreover one has

$$p^* \mathscr{V}(\mathfrak{g}; H) \subset \mathscr{V}(\mathfrak{n}; H), \tag{2.12}$$

where $p^* : \mathfrak{g}^* \to \mathfrak{n}^*$ denotes the restriction map of linear forms.

Proof. (1) Take a finite-dimensional generating subspace H_0 of H such that the associated graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_k M_k$ is finitely generated over $R = \mathrm{gr}\, A = \bigoplus_k R_k$. Then there exists a non-negative integer u such that $M = R(M_0 + \cdots + M_u)$, and so we have

$$M_k = R_k M_0 + \cdots + R_{k-u} M_u \qquad \text{for} \quad k \geqslant u. \tag{2.13}$$

By induction on k, one finds easily from this equality that

$$H_k \subset A_k H_u \qquad \text{for every} \quad k \geqslant 0, \tag{2.14}$$

where $A_k = A \cap U_k(\mathfrak{g})$ and $H_k = U_k(\mathfrak{g})\, H_0$ as before. Hence $H = AH_u$ is of finite type over A.

(2) For $A = U(\mathfrak{n})$, we have $R_m R_k = R_{k+m}$ $(k, m \geqslant 0)$ since $R = \mathrm{gr}\, A$ is just the symmetric algebra of $\mathfrak{n}$. In this case, the right-hand side of (2.13) becomes $R_{k-u} M_u$, so one obtains $M_k = R_{k-u} M_u$ and correspondingly,

$$H_k = A_{k-u} H_u \qquad \text{for each} \quad k \geqslant u. \tag{2.15}$$

This implies (2.10) and so (2.11) since, by the definition of $c(\cdot\,; H)$, $d(\cdot\,; H)$, one has

$$\dim A_{k-u}H_u = \frac{c(\mathfrak{n}; H)}{d(\mathfrak{n}; H)!}(k-u)^{d(\mathfrak{n}; H)} + (\text{the lower terms}),$$

$$\dim H_k = \frac{c(\mathfrak{g}; H)}{d(\mathfrak{g}; H)!}k^{d(\mathfrak{g}; H)} + (\text{the lower terms}),$$

for sufficiently large k.

Finally we prove (2.12). Since $H = AH_u$ with $A = U(\mathfrak{n})$, the associated variety $\mathscr{V}(\mathfrak{n}; H)$ is, by definition, given as

$$\mathscr{V}(\mathfrak{n}; H) = (\operatorname{Ann}_R N)^{\#}$$

through the graded R-module

$$N = \bigoplus_k N_k \qquad \text{with} \quad N_k = A_k H_u / A_{k-1} H_u.$$

Let D be a homogeneous element of $\operatorname{Ann}_R N$ of positive degree. It follows from (2.15) that $D \mid M_{k+u} = 0$ for $k \geqslant 0$. This implies that

$$D^{u+1}v = D(D^u v) = 0 \qquad \text{for every} \quad v \in M_0,$$

since $D^u v \in M_u$. Hence D^u annihilates the whole module $M = S(\mathfrak{g})\, M_0$, and one gets

$$\operatorname{Ann}_R N \subset \sqrt{\operatorname{Ann}_R M},$$

where $\sqrt{J} = \{D \in R \mid D^m \in J \text{ for some } m > 0\}$ denotes the radical of an ideal J of R. We thus conclude

$$\mathscr{V}(\mathfrak{n}; H) \supset \left(\sqrt{\operatorname{Ann}_R M}\right)^{\#} = (\operatorname{Ann}_R M)^{\#},$$

and the right-hand side clearly contains $p^*\mathscr{V}(\mathfrak{g}; H)$. This finishes the proof of our theorem. Q.E.D.

The following is a direct consequence of Theorem 2.2(2).

COROLLARY 2.3. *If a finitely generated $U(\mathfrak{g})$-module H has the good restriction to $U(\mathfrak{n})$, the Gelfand–Kirillov dimension $d(\mathfrak{g}; H)$ does not exceed* $\dim \mathfrak{n}$.

We now give some more consequences of Theorems 2.1 and 2.2, which will be used in Section 5 for our study of the finite multiplicity property for induced representations.

COROLLARY 2.4. *Let I be a right ideal of $U(\mathfrak{g})$ such that $I \neq U(\mathfrak{g})$. For a finitely generated $U(\mathfrak{g})$-module H, the factor space H/IH is finite-dimensional if $\mathscr{V}(\mathfrak{g}; H) \cap (\mathrm{gr}\, I)^{\#} = (0)$, where $\mathrm{gr}\, I = \bigoplus_k I_k/I_{k-1}$ with $I_k = U_k(\mathfrak{g}) \cap I$.*

Proof. Let $A = \mathbf{C}1 + I$ be the subalgebra of $U(\mathfrak{g})$ generated by I and the identity 1. In view of Theorems 2.1 and 2.2(1), one finds that the condition $\mathscr{V}(\mathfrak{g}; H) \cap (\mathrm{gr}\, I)^{\#} = (0)$ implies that H is of finite type over A. This gives the corollary. Q.E.D.

COROLLARY 2.5. *Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$, and let H be a finitely generated $U(\mathfrak{g})$-module satisfying the condition $\mathscr{V}(\mathfrak{g}; H) \cap \mathfrak{n}^{\perp} = (0)$. Then, the $\mathfrak{n}$-homology groups $H_k(\mathfrak{n}, H)$ $(k = 0, 1, \ldots)$ of H (see, e.g., [8] for the definition) are all finite-dimensional.*

Proof. It follows from Corollary 2.2 and Theorem 2.2(1) that H is of finite type over $U(\mathfrak{n})$. We get the assertion since the homology groups of finitely generated modules are always finite-dimensional (cf. [10, 1.6]). Q.E.D.

2.4. *A Generalization of Corollary* 2.4

Let I be a non-trivial right ideal of $U(\mathfrak{g})$. We denote by N_I the left normalizer of I in $U(\mathfrak{g})$:

$$N_I = \{D \in U(\mathfrak{g}) \mid DI \subset I\}. \tag{2.16}$$

For any $U(\mathfrak{g})$-module H, the factor space H/IH becomes an N_I-module.

We conclude this section with giving a generalization of Corollary 2.4, as follows.

PROPOSITION 2.4. *Let B be a subalgebra of N_I containing the identity element. Denote by $\mathrm{gr}\, I$ (resp. $\mathrm{gr}\, B$) the graded ideal (resp. graded subalgebra) of $S(\mathfrak{g})$ associated to I (resp. B). For a finitely $U(\mathfrak{g})$-module H, H/IH is of finite type over B whenever the variety $\mathscr{V}(\mathfrak{g}; H) \cap (\mathrm{gr}\, I)^{\#} \cap (\mathrm{gr}\, B)_+^{\#}$ reduces to (0). Here $(\mathrm{gr}\, B)_+$ denotes the maximal graded ideal of $\mathrm{gr}\, B$.*

This proposition actually includes Corollary 2.4 as a special case $B = \mathbf{C}1$.

An application of the proposition will be given in Section 3 for semisimple Lie algebras $\mathfrak{g}$.

Proof of Proposition 2.4. Let $\mathscr{F} = (H_k)_{k \geq 0}$, $H_k = U_k(\mathfrak{g})\, H_0$, be the increasing filtration of H defined by a finite-dimensional generating subspace H_0 of H. This $\mathscr{F}$ naturally induces filtrations of the subspace IH and of the quotient space H/IH, and we get the corresponding graded $(\mathrm{gr}\, N_I)$-modules

$$\mathrm{gr}(IH) = \bigoplus_k (H_k \cap IH)/(H_{k-1} \cap IH) \subset \mathrm{gr}(H; H_0) = \bigoplus_k H_k/H_{k-1}$$

and

$$\operatorname{gr}(H/IH) = \bigoplus_k ((H_k + IH)/IH)/((H_{k-1} + IH)/IH),$$

where $\operatorname{gr} N_I = \bigoplus_k (N_I \cap U_k(\mathfrak{g}))/(N_I \cap U_{k-1}(\mathfrak{g}))$. It is easy to see that

$$\operatorname{gr}(H/IH) \simeq \operatorname{gr}(H; H_0)/\operatorname{gr}(IH) \tag{2.17}$$

as $(\operatorname{gr} N_I)$-modules. Noting that $(\operatorname{gr} I) \operatorname{gr}(H; H_0) \subset \operatorname{gr}(IH)$, we get a surjective $(\operatorname{gr} N_I)$-homomorphism

$$T := \operatorname{gr}(H; H_0)/(\operatorname{gr} I) \operatorname{gr}(H; H_0) \to \operatorname{gr}(H/IH), \tag{2.18}$$

where T is a graded $S(\mathfrak{g})$-module.

If $\mathscr{V}(\mathfrak{g}; H) \cap (\operatorname{gr} I)^{\#} \cap (\operatorname{gr} B)^{\#}_{+} = (0)$, we deduce from Proposition 2.1 that T is of finite type over $\operatorname{gr} B$ since the associated variety of T is clearly contained in $\mathscr{V}(\mathfrak{g}; H) \cap (\operatorname{gr} I)^{\#}$. By (2.18), our $\operatorname{gr}(H/IH)$, which is a surjective image of T by a $(\operatorname{gr} N_I)$-homomorphism, is finitely generated over $\operatorname{gr} B$. This implies just as in the proof of Theorem 2.2(1) that H/IH is of finite type over B. Q.E.D.

3. Nilpotent Varieties in $\mathfrak{p}$ and Good Restriction of Harish-Chandra Modules

Until the end of Section 4, let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this section, applying the results of Section 2 we characterize, in relation to nilpotent varieties in $\mathfrak{p}$, subalgebras of $U(\mathfrak{g})$ to which all the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a symmetric decomposition of $\mathfrak{g}$. The main results here are stated in Theorems 3.1 and 3.2.

Although our principal interest lies in the applications to Harish-Chandra modules, we proceed here in a more general situation as much as possible

3.1. *Associated Varieties for $U(\mathfrak{g})$-Modules in $C(\mathfrak{k}, \mathscr{Z})$*

Let $\mathfrak{k}$ be any Lie subalgebra of $\mathfrak{g}$, and let $\mathscr{Z} = \mathscr{Z}(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. Set $B(\mathfrak{k}, \mathscr{Z}) = U(\mathfrak{k}) \mathscr{Z}(\mathfrak{g})$, and we consider as in 2.2 the category $C(\mathfrak{k}, \mathscr{Z}) := C(B(\mathfrak{k}, \mathscr{Z}))$ of locally $B(\mathfrak{k}, \mathscr{Z})$-finite, finitely generated $U(\mathfrak{g})$-modules.

A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is said to be *symmetrizing* if it is the set of fixed points of an involutive automorphism of $\mathfrak{g}$. In this case, the $U(\mathfrak{g})$-modules in $C(\mathfrak{k}, \mathscr{Z})$ will be called *Harish-Chandra* $(\mathfrak{g}, \mathfrak{k})$*-modules*. This category of Harish-Chandra modules is enjoying an essential role in representation theory of real semisimple Lie groups (see, e.g., [10, 20]).

On the other side, the category $C(\mathfrak{k}, \mathscr{Z})$ for a Borel subalgebra $\mathfrak{k}$ includes the highest weight modules (cf. [11, Chap. 7]).

We now study the associated varieties $\mathscr{V}(\mathfrak{g}; H)$ of $U(\mathfrak{g})$-modules H in $C(\mathfrak{k}, \mathscr{Z})$. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$, we regard $\mathscr{V}(\mathfrak{g}; H)$ as a variety in $\mathfrak{g}$. For a subset $\mathfrak{s}$ of $\mathfrak{g}$, let $\mathscr{N}(\mathfrak{s})$ denote the set of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{s}$.

LEMMA 3.1 (cf. [19, Cor. 5.13]). *Let $Q(\mathfrak{k}, \mathscr{Z}) = \operatorname{gr} B(\mathfrak{k}, \mathscr{Z})$ be the graded subalgebra of $S(\mathfrak{g})$ corresponding to $B(\mathfrak{k}, \mathscr{Z}) = U(\mathfrak{k})\, \mathscr{Z}(\mathfrak{g})$. Then the variety $Q(\mathfrak{k}, \mathscr{Z})^{\#}_{+}$ (see (2.1)) is contained in $\mathscr{N}(\mathfrak{p})$, and hence, by (2.6), it holds that*

$$\mathscr{V}(\mathfrak{g}; H) \subset Q(\mathfrak{k}, \mathscr{Z})^{\#}_{+} \subset \mathscr{N}(\mathfrak{p}) \tag{3.1}$$

for every $U(\mathfrak{g})$-module H in the category $C(\mathfrak{k}, \mathscr{Z})$. Here $\mathfrak{p} := \mathfrak{k}^{\perp}$ denotes the complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$.

Proof. By noting that $I(\mathfrak{g}) := \operatorname{gr} \mathscr{Z}(\mathfrak{g})$ is the subalgebra of $S(\mathfrak{g})$ consisting of all $(\operatorname{ad} \mathfrak{g})$-invariant elements of $S(\mathfrak{g})$, it is easy to see that

$$Q(\mathfrak{k}, \mathscr{Z}) \supset S(\mathfrak{k})\, I(\mathfrak{g}). \tag{3.2}$$

This implies that

$$Q(\mathfrak{k}, \mathscr{Z})^{\#}_{+} \subset \mathfrak{p} \cap I(\mathfrak{g})^{\#}_{+}.$$

One thus gets the lemma since $I(\mathfrak{g})^{\#}_{+}$ coincides with the nilpotent variety $\mathscr{N}(\mathfrak{g})$ of $\mathfrak{g}$ (see, e.g., [12, p. 370]). Q.E.D.

It should be noted that $\mathfrak{p}$ is an $(\operatorname{ad} \mathfrak{k})$-stable subspace of $\mathfrak{g}$.

For symmetrizing $\mathfrak{k}$ we can construct a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module $\tilde{H}$ whose associated variety $\mathscr{V}(\mathfrak{g}; \tilde{H})$ is exactly the whole nilpotent variety $\mathscr{N}(\mathfrak{p})$. To show this, we prepare the following

PROPOSITION 3.1. *Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{p} = (0)$ for $\mathfrak{p} = \mathfrak{k}^{\perp}$.*

(1) *One has $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as $(\operatorname{ad} \mathfrak{k})$-modules.*

(2) *The $U(\mathfrak{g})$-module*

$$\tilde{H} := U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^{K}_{+}) \tag{3.3}$$

lies in the category $C(\mathfrak{k}, \mathscr{Z})$, and its associated variety is described as

$$\mathscr{V}(\mathfrak{g}; \tilde{H}) = (S(\mathfrak{p})^{K})^{\#}_{+} \cap \mathfrak{p}. \tag{3.4}$$

Here $U(\mathfrak{g})^{K}_{+}$ (resp. $S(\mathfrak{p})^{K}$) denotes the set of elements D in $\mathfrak{g}\, U(\mathfrak{g})$ (resp. in $S(\mathfrak{p})$) such that $(\operatorname{ad} X) D = 0$ for all $X \in \mathfrak{k}$.

Proof. (1) The assumption $\mathfrak{k} \cap \mathfrak{p} = (0)$ implies that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, since the Killing form of $\mathfrak{g}$ is non-degenerate. Each component $\mathfrak{k}$, $\mathfrak{p}$ is $(\mathrm{ad}\, \mathfrak{k})$-stable. This proves (1).

(2) Noting that $\mathscr{Z} = \mathbf{C}1 + (\mathscr{Z} \cap \mathfrak{g}\, U(\mathfrak{g}))$ and that $\mathscr{Z} \cap \mathfrak{g}\, U(\mathfrak{g}) \subset U(\mathfrak{g})^K_+$, one sees easily that $\tilde{H}$ is in the category $C(\mathfrak{k}, \mathscr{Z})$.

Let $\mathrm{gr}\, \tilde{H}$ be the graded $S(\mathfrak{g})$-module associated to $\tilde{H}$ though the natural filtration $(U_n(\mathfrak{g}))_{n \geqslant 0}$ of $U(\mathfrak{g})$. It is given as $\mathrm{gr}\, \tilde{H} = S(\mathfrak{g})/J$ with

$$J := \bigoplus_n \left(U_n(\mathfrak{g}) \cap U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^K_+)\right) / \left(U_{n-1}(\mathfrak{g}) \cap U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^K_+)\right), \tag{3.5}$$

and hence we have by definition

$$\mathscr{V}(\mathfrak{g}; \tilde{H}) = J^{\#} \;\; (= \{X \in \mathfrak{g} \mid f(X) = 0 \;\; (\forall f \in J)\}). \tag{3.6}$$

Let us describe the graded ideal J. In view of (1), one gets from the Poincaré–Birkhoff–Witt theorem,

$$U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus \omega(S(\mathfrak{p})) \tag{3.7}$$

as vector spaces, where $\omega: S(\mathfrak{g}) \to U(\mathfrak{g})$ denotes the symmetrization, i.e., the unique linear isomorphism from $S(\mathfrak{g})$ onto $U(\mathfrak{g})$ characterized by $\omega(X^m) = X^m$ for $X \in \mathfrak{g}$ and integers $m \geqslant 0$.

Noting that ω commutes with the $(\mathrm{ad}\, \mathfrak{g})$-action, we easily deduce that

$$U(\mathfrak{g})\mathfrak{k} + U(\mathfrak{g})^K_+ = U(\mathfrak{g})\mathfrak{k} \oplus \omega(S(\mathfrak{p})^K_+). \tag{3.8}$$

This together with (3.7) yields

$$U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^K_+) = U(\mathfrak{g})\mathfrak{k} + \omega(S(\mathfrak{p}))\, \omega(S(\mathfrak{p})^K_+). \tag{3.9}$$

It follows from the Poincaré–Birkhoff–Witt theorem again, that the sum on the right-hand side of (3.9) is direct and that

$$\begin{aligned} &U_n(\mathfrak{g}) \cap (U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^K_+)) \\ &\quad = U_n(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{k} \oplus U_n(\mathfrak{g}) \cap (\omega(S(\mathfrak{p}))\, \omega(S(\mathfrak{p})^K_+)). \end{aligned} \tag{3.10}$$

This immediately implies that

$$J = S(\mathfrak{g})\mathfrak{k} + S(\mathfrak{p})\, S(\mathfrak{p})^K_+, \tag{3.11}$$

and thus we obtain (3.4) by (3.6), as desired. Q.E.D.

A nilpotent element $X \in \mathfrak{p}$ is called *normal* if there exists an element $T \in \mathfrak{k}$ and a non-zero complex number β such that $[T, X] = \beta X$. Let $\mathscr{N}_{\mathrm{nor}}(\mathfrak{p})$ denote the set of normal nilpotent elements in $\mathfrak{p}$.

We now arrive at

PROPOSITION 3.2. (1) *Let* $\mathfrak{k}$, $\mathfrak{p}=\mathfrak{k}^{\perp}$, *and let* $\tilde{H}$ *be as in Proposition* 3.1. *Then it holds that*

$$\mathcal{N}_{\mathrm{nor}}(\mathfrak{p}) \subset \mathscr{V}(\mathfrak{g}; \tilde{H}) \subset \mathcal{N}(\mathfrak{p}). \tag{3.12}$$

(2) *Assume* $\mathfrak{k}$ *to be symmetrizing. Then one has* $\mathfrak{k} \cap \mathfrak{p} = (0)$, *and the equalities hold in* (3.12). *Hence* $\tilde{H}$ *is a Harish-Chandra* $(\mathfrak{g}, \mathfrak{k})$*-module such that* $\mathscr{V}(\mathfrak{g}; \tilde{H}) = \mathcal{N}(\mathfrak{p})$.

Proof. (1) By (3.1), it is enough to show the first inclusion in (3.12). Let $X \in \mathcal{N}_{\mathrm{nor}}(\mathfrak{p})$ and take an element $T \in \mathfrak{k}$ such that $[T, X] = \beta X$ for some $\beta \neq 0$, $\in \mathbf{C}$. It then follows that

$$(\exp \mathbf{C}(\operatorname{ad} T))X = (\mathbf{C} \backslash (0))X.$$

Hence, any $(\operatorname{ad} \mathfrak{k})$-invariant polynomial f on $\mathfrak{p}$ is, by continuity, constant on $\mathbf{C}\,X$. We thus obtain $f(X) = f(0) = 0$ for all f in $S(\mathfrak{p})_+^K$. This together with (3.4) proves (3.12).

(2) Let σ be an involution of $\mathfrak{g}$ such that $\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$. Then one gets $\mathfrak{p} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, and so $\mathfrak{k} \cap \mathfrak{p} = (0)$. In this case, we see from [12, p. 370] that $\mathcal{N}_{\mathrm{nor}}(\mathfrak{p}) = \mathcal{N}(\mathfrak{p})$. Hence the equalities hold in (3.12). Q.E.D.

The following is an immediate consequence of Lemma 3.1 and Proposition 3.2(2).

COROLLARY 3.1 (See remark following Proposition 2.2). *Assume that* $\mathfrak{k}$ *is symmetrizing, and let* $\mathscr{V}_{B(\mathfrak{k}, \mathscr{Z})}$ *be the subset of* $Q(\mathfrak{k}, \mathscr{Z})_+^{\#}$ *defined in Proposition* 2.2, *where* $B(\mathfrak{k}, \mathscr{Z}) = U(\mathfrak{k})\,\mathscr{Z}(\mathfrak{g})$ *and* $Q(\mathfrak{k}, \mathscr{Z}) = \operatorname{gr} B(\mathfrak{k}, \mathscr{Z})$ *as before. Then one has*

$$\mathscr{V}_{B(\mathfrak{k}, \mathscr{Z})} = Q(\mathfrak{k}, \mathscr{Z})_+^{\#} = \mathcal{N}(\mathfrak{p}). \tag{3.13}$$

Remark. It is interesting to describe the associated varieties $\mathscr{V}(\mathfrak{g}; H)$ for important Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules H. We can achieve this for the discrete series of a semisimple Lie group by an elementary method based on Hotta and Parthasarathy's work [13] (see also [22]). The details will be discussed elsewhere.

3.2. *Characterization of Large Subalgebras Relative to* $B(\mathfrak{k}, \mathscr{Z})$

Let A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$, and let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$. Consider the following two conditions on A in relation to $\mathfrak{k}$.

(NPRO) $\mathcal{N}(\mathfrak{p}) \cap R_+^{\#} = (0)$, where $\mathfrak{p} = \mathfrak{k}^{\perp}$ and $R = \operatorname{gr} A$.

(ALKZ) A is large relative to $B(\mathfrak{k}, \mathscr{Z})$; i.e., all the $U(\mathfrak{g})$-modules H in the category $C(\mathfrak{k}, \mathscr{Z})$ have the good restriction to A. So, in this case, these modules H have the properties specified in 2.3.

Getting together the results in 2.2 and 3.1, we find a close relation between these conditions as follows.

THEOREM 3.1. *For A and $\mathfrak{k}$ as above, the condition* (NPRO) *always implies* (ALKZ). *Moreover, these two conditions are equivalent with each other if $R = \operatorname{gr} A$ is Noetherian and if $\mathfrak{k}$ is symmetrizing.*

Proof. The first statement follows from Theorem 2.1(1) and (3.1). Proposition 2.2 together with Corollary 3.1 gives the second one. Q.E.D.

As a special case, we obtain the following criterion.

THEOREM 3.2 ($\mathfrak{k}$: symmetrizing, $A = U(\mathfrak{n})$). *All the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction to a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ if and only if there does not exist any non-zero nilpotent element of $\mathfrak{g}$ orthogonal to $\mathfrak{k} + \mathfrak{n}$ with respect to the Killing form*:

$$\mathcal{N}((\mathfrak{k} + \mathfrak{n})^{\perp}) = \mathfrak{n}^{\perp} \cap \mathcal{N}(\mathfrak{p}) = (0). \tag{3.14}$$

By applying Proposition 2.4, one gets another consequence of condition (NPRO) as in

PROPOSITION 3.3. *Let $\mathfrak{k}$, A be as in Theorem 3.1, and let I be a proper, right ideal of $U(\mathfrak{g})$ such that $A/A \cap I$ is finite-dimensional. If* (NPRO) *is satisfied, the factor space H/IH is finitely generated as a $\mathscr{Z}(\mathfrak{g})$-module for every locally $\mathfrak{k}$-finite, finitely generated $U(\mathfrak{g})$-module H.*

Proof. We observe that $(\operatorname{gr} I)^{\#} = R_{+}^{\#}$ with $R = \operatorname{gr} A$, since $\dim A/A \cap I < \infty$. By (2.6), the associated variety $\mathcal{V}(\mathfrak{g}; H)$ of a locally $\mathfrak{k}$-finite $U(\mathfrak{g})$-module H is contained in $\mathfrak{p}$. Hence the condition (NPRO) assures that

$$\mathcal{V}(\mathfrak{g}; H) \cap (\operatorname{gr} I)^{\#} \cap (\operatorname{gr} \mathscr{Z})_{+}^{\#} = (0),$$

because $(\operatorname{gr} \mathscr{Z})_{+}^{\#} = \mathcal{N}(\mathfrak{g})$ as remarked in the proof of Lemma 3.1. Now Proposition 2.3 applies and we get the claim. Q.E.D.

4. Large Lie Subalgebras of a Real Semisimple Lie Algebra

Let $\mathfrak{g}_0$ be, through this section, a real semisimple Lie algebra, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ determined by an involution θ. We write $\mathfrak{h}$ $(\subset \mathfrak{g})$ for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript "0."

A Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is said to be *large* in $\mathfrak{g}_0$ if there exists an element $x \in \mathrm{Int}(\mathfrak{g}_0)$ for which the subalgebra $U(x \cdot \mathfrak{n})$ is large in $U(\mathfrak{g})$ relative to $B(\mathfrak{k}, \mathscr{Z}) = U(\mathfrak{k})\,\mathscr{Z}(\mathfrak{g})$ (see (ALKZ) in 3.2). This amounts to, by virtue of Theorem 3.2, a simple geometric condition:

$$(x \cdot \mathfrak{n})^{\perp} \cap \mathscr{N}(\mathfrak{p}) = (0) \qquad \text{for some} \quad x \in \mathrm{Int}(\mathfrak{g}_0). \tag{4.1}$$

Here $\mathrm{Int}(\mathfrak{g}_0)$ denotes the group of inner automorphisms of $\mathfrak{g}_0$. Notice that the largeness of a Lie subalgebra does not depend on the choice of a $\mathfrak{k}_0$, since such $\mathfrak{k}_0$'s are conjugate with each other by inner automorphisms.

In this section we first specify in 4.1–4.2 many of the large Lie subalgebras of $\mathfrak{g}_0$, and then in 4.3–4.4 we show that every quasi-spherical Lie subalgebra (cf. [3, 16]) is large in $\mathfrak{g}_0$.

4.1. *Two Kinds of Typical Large Lie Subalgebras*

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ be an Iwasawa decomposition of $\mathfrak{g}_0$. Here is the first important example of large Lie subalgebras of $\mathfrak{g}_0$.

PROPOSITION 4.1. *The maximal nilpotent Lie subalgebra $\mathfrak{u}_{m,0}$ is large in $\mathfrak{g}_0$.*

Proof. It follows from the Iwasawa decomposition of $\mathfrak{g}_0$ that $(\mathfrak{k} + \mathfrak{u}_m)^{\perp} = \mathfrak{a}_p$, which consists only of semisimple elements of $\mathfrak{g}$. Hence we have $\mathfrak{u}_m^{\perp} \cap \mathscr{N}(\mathfrak{p}) = \mathscr{N}((\mathfrak{k} + \mathfrak{u}_m)^{\perp}) = (0)$. Q.E.D.

The above proposition, together with Theorem 2.2, covers the results of Casselman and Osborne [9, Th. 2.3] and Joseph [14, II, 5.6] on the restriction of Harish-Chandra modules to $\mathfrak{u}_m$.

Second, let $\mathfrak{h}_0$ be any symmetrizing Lie subalgebra of $\mathfrak{g}_0$ defined by an involutive automorphism σ of $\mathfrak{g}_0$. Then there exists an inner automorphism y of $\mathfrak{g}_0$ such that $\sigma_y := y \circ \sigma \circ y^{-1}$ commutes with the Cartan involution θ. Let $\mathfrak{g}_0 = y \cdot \mathfrak{h}_0 \oplus \mathfrak{s}_0$ be the eigenspace decomposition of $\mathfrak{g}_0$ by σ_y. Take a maximal abelian subspace $\mathfrak{a}_{ps,0}$ of $\mathfrak{p}_0 \cap \mathfrak{s}_0$, and an element $X' \in \mathfrak{a}_{ps,0}$ which is regular in the sense: dim Ker(ad X') is minimal among the elements of $\mathfrak{a}_{ps,0}$. Then one has a Cartan decomposition of $\mathfrak{g}_0$ with respect to $y \cdot \mathfrak{h}_0$ as

$$\mathfrak{g}_0 = (\mathfrak{k}_0 + x'y \cdot \mathfrak{h}_0) \oplus \mathfrak{a}_{ps,0}, \tag{4.2}$$

where $x' = \exp(\mathrm{ad}\, X')$, and $\mathfrak{a}_{ps,0}$ is orthogonal to $\mathfrak{k}_0 + x'y \cdot \mathfrak{h}_0$ with respect to the Killing form. See [21, I, Lemma 1.9] for the proof of (4.2). We thus deduce

$$(x'y \cdot \mathfrak{h})^{\perp} \cap \mathscr{N}(\mathfrak{p}) = \mathscr{N}(\mathfrak{a}_{ps}) = (0), \tag{4.3}$$

because the elements of $\mathfrak{a}_{ps}$ are semisimple, and so this gives the second typical example of large Lie subalgebras.

PROPOSITION 4.2. *Any symmetrizing subalgebra* $\mathfrak{h}_0$ *is large in* $\mathfrak{g}_0$.

This allows us to deduce the finite multiplicity theorem [1] for the quasi-regular representation on $L^2(G/H)$, associated to a semisimple symmetric space G/H.

4.2. *Inheritance of the Largeness by Parabolic Induction*

Let $\mathfrak{q}_0$ be any parabolic subalgebra of $\mathfrak{g}_0$, and let $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$ with $\mathfrak{l}_0 = \mathfrak{q}_0 \cap \theta\mathfrak{q}_0$, be its Levi decomposition. Since the Levi component $\mathfrak{l}_0 = (\mathfrak{k} \cap \mathfrak{l}_0) + (\mathfrak{p} \cap \mathfrak{l}_0)$ is reductive, one can define large Lie subalgebras of $\mathfrak{l}_0$ just in the same way.

The largeness of Lie subalgebras is preserved by parabolic induction.

LEMMA 4.1. *If* $\mathfrak{h}_0$ *is a large Lie subalgebra of* $\mathfrak{l}_0$, *the semidirect product Lie subalgebra* $\mathfrak{h}_0 + \mathfrak{u}_0$ *is large in* $\mathfrak{g}_0$.

Proof. We can take an inner automorphism z of $\mathfrak{l}_0$ such that $\mathcal{N}((\mathfrak{k} \cap \mathfrak{l} + z \cdot \mathfrak{h})^\perp \cap \mathfrak{l}) = (0)$. It follows that

$$(\mathfrak{k} \cap \mathfrak{l} + z \cdot \mathfrak{h})^\perp \cap \mathfrak{l} = (\mathfrak{k} + z \cdot (\mathfrak{h} + \mathfrak{u}))^\perp,$$

since $(\mathfrak{k} + \mathfrak{u})^\perp = \mathfrak{p} \cap \mathfrak{l}$ and $z \cdot \mathfrak{u} = \mathfrak{u}$. One thus finds that $\mathfrak{h}_0 + \mathfrak{u}_0$ is large in $\mathfrak{g}_0$. Q.E.D.

Thanks to the above lemma, we can generalize Proposition 4.2 to

PROPOSITION 4.3 (cf. [21]). *Let* $\mathfrak{h}_0$ *be a symmetrizing subalgebra of the Levi factor* $\mathfrak{l}_0$ *of a parabolic subalgebra* $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$. *Then* $\mathfrak{h}_0 + \mathfrak{u}_0$ *is large in* $\mathfrak{g}_0$.

This proposition actually contains Proposition 4.2 as a special case $\mathfrak{q}_0 = \mathfrak{g}_0$.

Using this proposition, we can recover our finite multiplicity theorems for induced representations of semisimple Lie groups, given in [21, I]. See 5.4 for the details.

4.3. *Quasi-Sperical Lie Subalgebras*

Let $\mathfrak{q}_{m,0} = \mathfrak{m}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ be a minimal parabolic subalgebra of $\mathfrak{g}_0$, where $\mathfrak{m}_0$ denotes the centralizer of $\mathfrak{a}_{p,0}$ in $\mathfrak{k}_0$. We say that a Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is *quasi-spherical* if there exists a $z \in \mathrm{Int}(\mathfrak{g}_0)$ such that $z \cdot \mathfrak{n}_0 + \mathfrak{q}_{m,0} = \mathfrak{g}_0$. This is equivalent to saying that, if G is a connected Lie group with Lie algebra $\mathfrak{g}_0$, the analytic subgroup of G corresponding to $\mathfrak{n}_0$ has an open orbit on the maximal flag variety G/Q_m, where Q_m denotes a minimal parabolic subgroup of G.

It is easy to verify that the large Lie subalgebras specified in 4.1–4.2 are all quasi-spherical.

We now show the following theorem, which is the principal result of this section.

THEOREM 4.1. *Quasi-spherical Lie subalgebras are always large in* $\mathfrak{g}_0$.

Remark. One can see from Theorem 3.2, coupled with a recent result of Bien and Oshima, that the converse is also true in the above theorem if $\mathfrak{n}_0$ is algebraic in $\mathfrak{g}_0$, i.e., $\mathfrak{n}_0$ is the Lie algebra of an algebraic subgroup N of G, where G is a semisimple algebraic group with Lie algebra $\mathfrak{g}_0$.

In fact, it is easy to deduce from our Theorem 3.2 that, if $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, the induced representations $\mathrm{Ind}_N^G(\eta)$ have the finite multiplicity property for all finite-dimensional N-representations η (see 5.4; for this, $\mathfrak{n}_0$ need not to be algebraic). A result of Bien and Oshima assures that, under the above assumption, these representations $\mathrm{Ind}_N^G(\eta)$ are of finite multiplicity only when $\mathfrak{n}_0$ is quasi-spherical.

4.4. *Proof of Theorem* 4.1

For the proof we prepare two elementary lemmas. Let V be any complex vector space of finite dimension, let Z be a subspace of V, and let F be a closed cone of V^*. We say that a sequence $(Z_j)_{j=0,1,\dots}$ of subspaces of V approximates Z if there exists a subset $\mathcal{O}$ of Z which generates Z as a vector space and which satisfies the condition

$$\mathcal{O} \subset \overline{\bigcup_{j \in S} Z_j} \tag{4.4}$$

for any infinite set S of positive integers, where $\bar{D}$ stands for the closure of a subset D of V.

LEMMA 4.2. *Assume that* $(Z_j)_j$ *approximates a subspace* Z *and that* $F \cap Z^\perp = (0)$ *for a closed cone* $F \subset V^*$. *Then* $F \cap Z_j^\perp = (0)$ *holds for sufficiently large* j.

Proof. Suppose that the claim is not true. Then, by taking a subsequence (j_r) of (j) if necessary, we can assume that there exists a sequence $(\lambda_j)_{j=0,1,\dots}$ of non-zero unit vectors (with respect to any given inner product on V^*) $\lambda_j \in F \cap Z_j^\perp$ that has a limit $\lambda = \lim_{j\to\infty} \lambda_j$. Since F is closed, λ is a non-zero vector in F.

Let X be any element of $\mathcal{O}$ in (4.4). Then there exists an increasing sequence $j_1 < j_2 < \cdots$ of positive integers and elements $X_s \in Z_{j_s}$ $(s = 1, 2, \dots)$ such that $X = \lim_{s\to\infty} X_s$. We thus deduce

$$\langle \lambda, X \rangle = \lim_{s\to\infty} \langle \lambda_{j_s}, X_s \rangle = 0$$

since $\lambda_{j_s} \in Z_{j_s}^\perp$, and hence $\lambda \in F \cap Z^\perp = (0)$. This is a contradiction. Q.E.D.

LEMMA 4.3. *Let B be an hermitian linear operator on V, and let $V = \bigoplus_\mu V(\mu)$ denote the eigenspace decomposition of V by B, where $V(\mu) = \{v \in V \mid Bv = \mu v\}$. For a real number r, set $V_r^+ = \bigoplus_{\mu > r} V(\mu)$ and $V_r^- = \bigoplus_{\mu \leqslant r} V(\mu)$. Then, if W is a subspace of V such that $V = W + V_r^-$, the sequence of subspaces $(\exp jB) W$ $(j = 0, 1, 2, \ldots)$ approximates V_r^+.*

Proof. Let $v \in V(\mu)$ with $\mu > r$. By the assumption there exists an element $w \in W$ such that $v - w \in V_r^-$. It then follows that $v = \lim_{j \to \infty} e^{-j\mu}(\exp jB) w$. This proves the lemma. Q.E.D.

We are now ready to prove Theorem 4.1. Let $\mathfrak{n}_0$ be a quasi-spherical Lie subalgebra of $\mathfrak{g}_0$. We can and do assume that $\mathfrak{n}_0 + \mathfrak{q}_{m,0} = \mathfrak{g}_0$. With the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}_p$ in mind, take an element $X \in \mathfrak{a}_{p,0}$ such that the linear operator $\operatorname{ad} X$, restricted to $\mathfrak{u}_m$, is negative-definite. Thanks to Lemma 4.3, the sequence of subspaces $\mathfrak{n}_j := (\exp(\operatorname{ad} jX))\mathfrak{n}$ $(j = 1, 2, \ldots)$ approximates $\theta\mathfrak{u}_m$. Noting that

$$\mathfrak{g} = \theta\mathfrak{u}_m \oplus \mathfrak{q}_m \qquad \text{and} \qquad \mathcal{N}(\mathfrak{p}) \cap (\theta\mathfrak{u}_m)^\perp = \mathcal{N}(\mathfrak{a}_p) = (0),$$

we conclude from Lemma 4.2 that $\mathcal{N}(\mathfrak{p}) \cap \mathfrak{n}_j^\perp = (0)$ for sufficiently large j. Thus $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, and we complete the proof of Theorem 4.1.

5. FINITE MULTIPLICITY THEOREMS FOR INDUCED REPRESENTATIONS

Let G be any connected Lie group with Lie algebra $\mathfrak{g}_0$ (not necessarily semisimple), and let A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$, where $\mathfrak{g}$ stands for, as in Section 4, the complexification of $\mathfrak{g}_0$. Following the idea of induced representations, we can associate, to any given Fréchet A-module E, an analytically induced G- and $U(\mathfrak{g})$-module $\Gamma(G \uparrow A; E)$ (see 5.1).

This section makes clear what we can know about these modules $\Gamma(G \uparrow A; E)$ by applying our results in Sections 2–4 (see Theorems 5.1 and 5.2). Moreover, for semisimple G, we develop and simplify our previous work [21] on the finiteness of multiplicities in induced representations by making use of the associated varieties of Harish-Chandra modules (see Theorems 5.3–5.5).

5.1. *Analytically Induced Modules $\Gamma(G \uparrow A; E)$ and $\mathscr{A}(G; \eta)$*

We begin with the precise definition of our induced modules. Let A be as above, and let E be an A-module with Fréchet space structure on which the elements of A act as continuous linear operators.

We then define $\Gamma = \Gamma(G \uparrow A; E)$ to be the space of all E-valued, real analytic functions f on G satisfying

$$R_D f(x) = {}^tD \cdot f(x) \tag{5.1}$$

for $D \in {}'A$ and $x \in G$. Here $D \to {}'D$ is the principal anti-automorphism of $U(\mathfrak{g})$ (see 2.2), and $D \to R_D$ identifies $U(\mathfrak{g})$ with the algebra of left invariant differential operators on G. The group G acts on Γ by left translation L:

$$L_g f(x) = f(g^{-1}x) \qquad (g \in G). \tag{5.2}$$

The $U(\mathfrak{g})$-action on Γ, gained by differentiation, will be denoted again by L. We call $(L, \Gamma(G \uparrow A; E))$ the *G-representation* or *$U(\mathfrak{g})$-module analytically induced from E.*

If (η, E) is a smooth Fréchet representation (cf. [21, I, 2.1]) of a closed subgroup N of G, the real analytic functions $f: G \to E$ such that

$$f(gn) = \eta(n)^{-1} f(g) \qquad \text{for} \quad (n, g) \in N \times G,$$

form a G-submodule, say $\mathscr{A}(G; \eta)$, of $\Gamma(G \uparrow U(\mathfrak{n}); E)$. Here $\mathfrak{n}$ is the complexified Lie algebra of N, and E is viewed as a $U(\mathfrak{n})$-module through differentiation. In this sense our $\Gamma(G \uparrow A; E)$'s include the group theoretical (analytically) induced modules $\mathscr{A}(G; \eta)$.

Now let H be a $U(\mathfrak{g})$-module. We discuss $U(\mathfrak{g})$-homomorphisms from H to $\Gamma = \Gamma(G \uparrow A; E)$ and especially the intertwining number

$$I_{U(\mathfrak{g})}(H, \Gamma) := \dim \operatorname{Hom}_{U(\mathfrak{g})}(H, \Gamma). \tag{5.3}$$

When H is irreducible, $I_{U(\mathfrak{g})}(H, \Gamma)$ gives the multiplicity of H in Γ as $U(\mathfrak{g})$-submodules.

Fix an element $x \in G$. If T is a $U(\mathfrak{g})$-homomorphism from H to Γ,

$$\iota_x(T)(v) := (Tv)(x) \qquad (v \in H) \tag{5.4}$$

gives rise to a linear map $\iota_x(T)$ from H to E. It is easily verified that $\iota_x(T)$ commutes with the actions of $xA := \mathrm{Ad}(x)A \subset U(\mathfrak{g})$ as

$$\iota_x(T) \circ D = (x^{-1}D) \circ \iota_x(T) \tag{5.5}$$

for all $D \in xA$, where $x^{-1}D = \mathrm{Ad}(x)^{-1}D$. Moreover, $\iota_x(T) = 0$ implies $T = 0$, since Tv $(v \in H)$ are real analytic functions on connected G.

We have thus obtained a half part of the Frobenius reciprocity for induced modules, as follows.

PROPOSITION 5.1. *Let H, $\Gamma = \Gamma(G \uparrow A; E)$ and let $x \in G$ be as above. The assignment $T \to \iota_x(T)$ defined in* (5.4) *gives an injective linear map*

$$\iota_x: \operatorname{Hom}_{U(\mathfrak{g})}(H, \Gamma) \hookrightarrow \operatorname{Hom}_{xA}(H, E_x), \tag{5.6}$$

where E_x stands for the Fréchet space E viewed as an (xA)-module by $D \cdot e = (x^{-1}D)e$ $(e \in E)$.

This proposition allows us to give in the succeeding subsections criteria for the finiteness of intertwining numbers $I_{U(\mathfrak{g})}(H, \Gamma)$ by means of the associated varieties of H and A.

5.2. *Finite Multiplicity Criteria*, I

First, observe that the vector space $\mathrm{Hom}_{xA}(H, E_x)$ in (5.6) is finite-dimensional if both A-module E and factor space $H/I_x H$ with $I_x := (\mathrm{Ann}_{xA} E_x)\, U(\mathfrak{g})$ are. Corollary 2.4 together with Proposition 5.1 gives the following finiteness criterion, which is the first important result of this section.

THEOREM 5.1. *Let H be a finitely generated $U(\mathfrak{g})$-module. The intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ from H to an analytically induced $U(\mathfrak{g})$-module $\Gamma = \Gamma(G \uparrow A; E)$ is finite whenever two conditions,*

$$\dim E < \infty, \tag{5.7}$$

and

$$\mathscr{V}(\mathfrak{g}; H) \cap x^{-1} \cdot R_+^{\#} = (0) \qquad \text{for some} \quad x \in G, \tag{5.8}$$

are satisfied. Here $\mathscr{V}(\mathfrak{g}; H)$ is the associated variety of H, $R_+^{\#}$ with $R = \mathrm{gr}\, A$ is the algebraic variety of $\mathfrak{g}^*$ defined in 2.1, and G acts on $\mathfrak{g}^*$ through the coadjoint representation.

Proof. We show that $H/I_x H$ is finite-dimensional under assumptions (5.7) and (5.8). In fact, by (5.7), the annihilator $\mathrm{Ann}_{xA} E_x \subset xA$ is of finite codimension in xA. This implies that the radical of the associated graded subalgebra $\mathrm{gr}(\mathrm{Ann}_{xA} E_x) \subset xR$ coincides with xR_+, the maximal homogeneous ideal of xR. Noting that $I_x \supset \mathrm{Ann}_{xA} E_x$, one sees

$$(\mathrm{gr}\, I_x)^{\#} \subset (xR_+)^{\#} = x^{-1} \cdot R_+^{\#}. \tag{5.9}$$

Hence condition (5.8) guarantees that $\dim H/I_x H < \infty$, by Corollary 2.4. The theorem now follows from Proposition 5.1, as mentioned above.

Q.E.D.

For a subalgebra B, $\ni 1$, of $U(\mathfrak{g})$, let $C(B)$ be, as in 2.2, the category of locally B-finite, finitely generated $U(\mathfrak{g})$-modules. The above theorem together with (2.6) immediately gives

COROLLARY 5.1. *Let $R = \mathrm{gr}\, A$, $Q = \mathrm{gr}\, B$ be the graded subalgebras of $S(\mathfrak{g})$ associated to subalgebras $A, B \subset U(\mathfrak{g})$, respectively. If there exists an element $x \in G$ such that $R_+^{\#} \cap x \cdot Q_+^{\#} = (0)$, the intertwining number $I_{U(\mathfrak{g})}(H, \Gamma(G \uparrow A; E))$ is finite for every $U(\mathfrak{g})$-module H in $C(B)$ and for every A-module E of finite dimension.*

5.3. *Estimation of the Multiplicities*

Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$, and let H be a $U(\mathfrak{g})$-module in the category $C(B)$ with $B = U(\mathfrak{k})$. Take a finite-dimensional, B-stable, generating subspace H_0. By noting that B is generated by 1 and $\mathfrak{k}$ as algebra, it is easy to see that the subspaces $H_k = U_k(\mathfrak{g})\, H_0$ $(k = 0, 1, ...)$ are all B-stable. Hence the corresponding graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_{k \geqslant 0} M_k$, with $M_k = H_k/H_{k-1}$, admits a natural B-module structure. Write this B-action on M by

$$B \times M \ni (D, v) \to D \diamond v \in M,$$

in order to distinguish it from the original $S(\mathfrak{g})$-action. One finds from the definition,

$$X \diamond Dv - D(X \diamond v) = ((\mathrm{ad}\, X) D) v \tag{5.10}$$

for $X \in \mathfrak{k}$ and $D \in S(\mathfrak{g})$.

With (5.6) in mind, we can give, by using this $(S(\mathfrak{g}), B)$-module M, an upper bound of the intertwining number $I_A(H, E) = \dim \mathrm{Hom}_A(H, E)$ as in

PROPOSITION 5.2. *Let H, $B = U(\mathfrak{k})$ be as above, and let A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$. One has for any A-module E,*

$$I_A(H, E) \leqslant \sum_{k=0}^{\infty} I_{A \cap B}(M_k/((A \cap B) \diamond (R_+ M)_k), E), \tag{5.11}$$

where $R = \mathrm{gr}\, A$ and $(R_+ M)_k := R_+ M \cap M_k$.

This together with (5.6) immediately gives the following theorem.

THEOREM 5.2. *The intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ from H in $C(B)$, $B = U(\mathfrak{k})$, to an analytically induced $U(\mathfrak{g})$-module $\Gamma = \Gamma(G \uparrow A; E)$ is bounded by*

$$\min_{x \in G} \left\{ \sum_{k=0}^{\infty} I_{xA \cap B}(M_k/((xA \cap B) \diamond ((xR_+) M)_k), E_x) \right\}. \tag{5.12}$$

Remarks. (1) By setting $\mathfrak{k} = (0)$, or $B = \mathbf{C}1$, one finds that this theorem recovers Theorem 5.1. In fact, in this case (5.12) turns out to be

$$\dim E \times \{\min_{x \in G} (\dim(M/(xR_+)M))\},$$

which is finite under assumptions (5.7) and (5.8) (see Proposition 2.1).

(2) If $A = U(\mathfrak{n})$ for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, one finds from the Poincaré–Birkhoff–Witt theorem that $xA \cap B = U(x \cdot \mathfrak{n} \cap \mathfrak{k})$ with $x \cdot \mathfrak{n} = \mathrm{Ad}(x)\mathfrak{n}$. Hence, in view of (5.10) we have

$$(xA \cap B) \diamond ((xR_+)M)_k = ((xR_+)M)_k = ((x \cdot \mathfrak{n})M)_k$$

in (5.12).

(3) The above theorem allows us to show $I_{U(\mathfrak{g})}(H, \Gamma) < \infty$ even for infinite-dimensional E's (cf. Theorems 5.4 and 5.5).

Proof of Proposition 5.2. For each integer $k \geqslant 0$, let J^k be the subspace of $J := \mathrm{Hom}_A(H, E)$ consisting of the A-homomorphisms $T: H \to E$ which are identically zero on H_k. One obtains a decreasing filtration of J:

$$J \supset J^0 \supset \cdots \supset J^{k-1} \supset J^k \supset \cdots, \qquad \bigcap_k J^k = (0). \tag{5.13}$$

If $T \in J^{k-1}$, the restriction $T \mid H_k$ of T to H_k induces $A \cap B$-homomorphism from $M_k = H_k / H_{k-1}$ to E. By definition, $T \mid H_k \equiv 0$ if and only if $T \in J^k$. Thus we have

$$r_k : J^{k-1}/J^k \hookrightarrow \mathrm{Hom}_{A \cap B}(M_k, E) \tag{5.14}$$

as vector spaces.

We now show that the $(A \cap B)$-module map $r_k(T + J^k)$ is identically zero on $(R_+ M)_k = \sum_{1 \leqslant j \leqslant k} R_j M_{k-j}$ for any $T \in J^{k-1}$. In fact, if $v \in R_j M_{k-j}$ $(j \geqslant 1)$, there exists a $D \in A_j$ and a $w \in H_{k-j}$ such that $v = Dw + H_{k-1}$. Noting that T is an A-homomorphism identically zero on H_{k-1}, we deduce

$$r_k(T + J^k)v = T(Dw) = DT(w) = 0,$$

since $w \in H_{k-j} \subset H_{k-1}$. Hence our $(A \cap B)$-homomorphism is zero on $(A \cap B) \diamond (R_+ M)_k$.

In view of (5.14) we conclude

$$\dim J^{k-1}/J^k \leqslant I_{A \cap B}(M_k / ((A \cap B) \diamond (R_+ M)_k), E). \tag{5.15}$$

This together with (5.13) gives (5.11). Q.E.D.

5.4. *Finite Multiplicity Criteria*, II: *The Case of Semisimple Lie Groups*

Now assume G to be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. In this subsection we aplly the results of 5.2 and 5.3 to Harish-Chandra modules for G.

By keeping the notation in Section 4, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with $\mathfrak{k}_0 = \mathrm{Lie}(K)$ denote a Cartan decomposition of $\mathfrak{g}_0 = \mathrm{Lie}(G)$. Let H be a Harish-Chandra

$(\mathfrak{g}, \mathfrak{k})$-module (see 3.1). Assume that the compact group K acts on H in such a way as

$$\dim\{Kv\} < \infty,$$

and

$$(d/dt)_{t=0} (\exp tX)v = Xv$$

for $v \in H$ and $X \in \mathfrak{k}_0$, where $\{Kv\}$ stands for the K-submodule of H generated by v. Such an H is called a *Harish-Chandra* $(\mathfrak{g}, K)$-*module*. Observe that, since our K is connected, the above two conditions assure the compatibility of $\mathfrak{g}$ and K actions,

$$k \cdot Xv = (k \cdot X) \cdot kv$$

for $k \in K$ and $X \in \mathfrak{g}$, where $k \cdot X = \mathrm{Ad}(k)X$.

We note that, if a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module H appears in some $\Gamma = \Gamma(G \uparrow A; E)$ as a $U(\mathfrak{g})$-submodule, H necessarily has the $(\mathfrak{g}, K)$-module structure inherited from Γ. A fundamental theorem of Harish-Chandra says that the (irreducible) Harish-Chandra $(\mathfrak{g}, K)$-modules correspond to the (irreducible) admissible representations of G, by passing to the K-finite part (see, e.g., [20, Chap. 8]). For these two reasons we concentrate on the $(\mathfrak{g}, K)$-modules from now on.

DEFINITION. Let $\Gamma = \Gamma(G \uparrow A; E)$ and $\mathscr{A} = \mathscr{A}(G; \eta)$ be the induced G- and $U(\mathfrak{g})$-modules defined in 5.1. We say that Γ (resp. $\mathscr{A}$) has the *finite multiplicity property* if the intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ (resp. $I_{U(\mathfrak{g})}(H, \mathscr{A})$) is finite for every Harish-Chandra $(\mathfrak{g}, K)$-module H.

Remarks. (1) Any $U(\mathfrak{g})$-homomorphism from H to Γ or to $\mathscr{A}$ commutes with the K-actions by virtue of the connectedness of K.

(2) One may define, just in the same way, the G- and $U(\mathfrak{g})$-modules $\Gamma^\infty = \Gamma^\infty(G \uparrow A; E)$ and $C^\infty(G; \eta)$ induced in the C^∞-context, which include Γ and $\mathscr{A}$ as submodules, respectively. Nevertheless concerning the $U(\mathfrak{g})$-homomorphisms from Harish-Chandra $(\mathfrak{g}, K)$-modules H, no difference is caused by considering these bigger Γ^∞ and $C^\infty(G; \eta)$.

In fact, every $\mathscr{Z}(\mathfrak{g})\, U(\mathfrak{k})$-finite function in Γ^∞ is real analytic on G by virtue of the regularity theorem of elliptic differential operators (see [21, I, 2.2]). Hence any $U(\mathfrak{g})$-module map from H to Γ^∞ (resp. to $C^\infty(G; \eta)$) carries H into Γ (resp. into $\mathscr{A}$).

(3) When η is a finite-dimensional unitary representation of a closed subgroup N, the assignment $H \to I_{U(\mathfrak{g})}(H, \mathscr{A})$ gives an upper bound of the

multiplicity function for G-representation $L^2\text{-Ind}_N^G(\eta)$ unitarily induced from η. Here H runs over the Harish-Chandra $(\mathfrak{g}, K)$-modules associated with irreducible unitary representations of G. See [21, I, Section 3] for the details.

Here is our first application to semisimple group G, of the general results in 5.2–5.3, which follows immediately from Corollaries 3.1 and 5.1.

PROPOSITION 5.3. *The induced module $\Gamma(G \uparrow A; E)$ has the finite multiplicity property for any finite-dimensional A-module E, if there exists an $x \in G$ for which $\mathcal{N}(\mathfrak{p}) \cap x \cdot (\mathrm{gr}\, A)_+^\# = (0)$* (cf. (NPRO) in 3.2). *Here $\mathcal{N}(\mathfrak{p})$ is the totality of nilpotent elements in $\mathfrak{p}$.*

As a special case, we gain

COROLLARY 5.2. *If $\mathfrak{n}_0$ is a large Lie subalgebra of $\mathfrak{g}_0$ (see Sect. 4), the conclusion of Proposition 5.3 is true for $A = U(\mathfrak{n})$ with $\mathfrak{n} = \mathfrak{n}_0 \otimes_{\mathbf{R}} \mathbf{C}$.*

In view of the large Lie subalgebras specified in Section 4, one may realize that this corollary has a number of applications.

Remark. For quasi-spherical Lie subalgebras $\mathfrak{n}_0$ (see 4.3), Bien and Oshima got a result similar to the above corollary. But our method here is completely different from theirs.

Now let (η, E) be a smooth Fréchet representation of a closed subgroup N of G, and consider the induced module $\mathscr{A}(G; \eta)$. For a Harish-Chandra $(\mathfrak{g}, K)$-module H, take a finite-dimensional, K-stable generating subspace H_0 of H. Then the associated graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_k M_k$ has a natural K-module structure.

We can estimate the intertwining number $I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$ from H to $\mathscr{A}(G; \eta)$:

THEOREM 5.3. *For each $x \in G$, one has the inequality*

$$I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta)) \leqslant \sum_{k=0}^{\infty} I_{K \cap xNx^{-1}}(M_k/((x \cdot \mathfrak{n})M)_k, E_x), \qquad (5.16)$$

where $((x \cdot \mathfrak{n})M)_k = M_k \cap (x \cdot \mathfrak{n})M$, and (η_x, E_x) is the representation of xNx^{-1} on E defined by $\eta_x(xnx^{-1}) = \eta(n)$ $(n \in N)$.

Proof. Recall that $\mathscr{A}(G; \eta)$ is a $U(\mathfrak{g})$-submodule of $\Gamma(G \uparrow U(\mathfrak{n}); E)$ (see 5.1). So, if the group $K \cap xNx^{-1}$ is connected, (5.16) follows directly from Theorem 5.2 with remark (2) of that theorem in mind. Even for non-connected $K \cap xNx^{-1}$, it is an easy task to modify the arguments in 5.2–5.3 and to deduce (5.16), by noting that the group N as well as $U(\mathfrak{n})$ acts on E. Q.E.D.

The above theorem enables us to deduce criteria for the finiteness of intertwining numbers $I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$, which are applicable even to infinite-dimensional (η, E)'s.

To be specific, fix an $x \in G$, and let Π denote the set of equivalence classes of irreducible finite-dimensional representations of $K \cap xNx^{-1}$. Then the locally finite $(K \cap xNx^{-1})$-module $M/(x \cdot \mathfrak{n})M = \bigoplus_k M_k/((x \cdot \mathfrak{n})M)_k$ is decomposed into a direct sum of the irreducibles as

$$M/(x \cdot \mathfrak{n})M \simeq \bigoplus_{\gamma \in \Pi} [m_\gamma] V_\gamma,$$

where V_γ is an irreducible $(K \cap xNx^{-1})$-module of class γ, and m_γ denotes the multiplicity of γ in $M/(x \cdot \mathfrak{n})M$.

One finds that (5.16) is rewritten as

$$I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta)) \leqslant \sum_{\gamma \in \Pi} m_\gamma I_{K \cap xNx^{-1}}(V_\gamma, E_x). \tag{5.17}$$

The sum on the right-hand side is finite if and only if there exists a finite subset F of Π for which

$$m_\gamma = 0 \quad \text{or} \quad I_{K \cap xNx^{-1}}(V_\gamma, E_x) = 0 \quad \text{for} \quad \gamma \notin F, \tag{5.18}$$

and

$$m_\gamma I_{K \cap xNx^{-1}}(V_\gamma, E_x) < \infty \qquad \text{for} \quad \gamma \in F. \tag{5.19}$$

The above discussion coupled with Corollary 2.2 leads us to the following

THEOREM 5.4. *Under the above notation, the intertwining number $I_{I(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$ from a Harish-Chandra module H to an induced $U(\mathfrak{g})$-module $\mathscr{A}(G; \eta)$ takes finite value if there exists an $x \in G$ such that*

$$\mathscr{V}(\mathfrak{g}; H) \cap (x \cdot \mathfrak{n})^\perp = (0), \tag{5.20}$$

and that

$$I_{K \cap xNx^{-1}}(V_\gamma, E_x) < \infty \text{ holds} \tag{5.21}$$

for every irreducible constituent V_γ of $M/(x \cdot \mathfrak{n})M$. Here $M = \mathrm{gr}(H; H_0)$, and $\mathscr{V}(\mathfrak{g}; H)$ denotes the associated variety of H.

Proof. By Corollary 2.2, condition (5.20) assures that $M/(x \cdot \mathfrak{n})M$ is finite-dimensional. So, let F be the finite subset of Π consisting of the irreducible constituents of $M/(x \cdot \mathfrak{n})M$. In view of (5.21), one finds that two conditions (5.18) and (5.19) are fulfilled for this F. This gives the theorem. Q.E.D.

From the above theorem, we immediately deduce a criterion for $\mathscr{A}(G;\eta)$ to be of multiplicity finite, as follows.

THEOREM 5.5. *Let N be a closed subgroup of G whose Lie algebra $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, and take an element $x \in G$ such that $(x\cdot\mathfrak{n})^{\perp} \cap \mathscr{N}(\mathfrak{p}) = (0)$. Then, for a smooth Fréchet representation (η, E) of N, the induced module $\mathscr{A}(G;\eta)$ has the finite multiplicity property if so is the restriction of η to the compact subgroup $x^{-1}Kx \cap N$.*

This theorem extends one of the principal results in our previous work [21, I, Th. 2.12], where we studied the case of semidirect product large Lie subalgebras $\mathfrak{n}_0 = \mathfrak{h}_0 + \mathfrak{u}_0$ specified in Proposition 4.3, through the theory of (K, N)-spherical functions. Interesting applications are found in [21, II] for reduced generalized Gelfand–Graev representations.

5.5. *Relation to K-Harmonic Polynomials on* $\mathfrak{p}$

We conclude this article by relating the $(K \cap xNx^{-1})$-module $M/(x\cdot\mathfrak{n})M$ in Theorems 5.3 and 5.4 to K-harmonic polynomials on $\mathfrak{p}$.

As in Section 3, regard the elements of $S(\mathfrak{p})$ as polynomial functions on $\mathfrak{p}$ through the Killing form of $\mathfrak{g}$. An element $f \in S(\mathfrak{p})$ is called *K-harmonic* if f is annihilated by every $\mathrm{Ad}(K)$-invariant, constant coefficient differential operator on $\mathfrak{p}$ without constant term. Let $\mathscr{H}(\mathfrak{p})$ denote the totality of K-harmonic polynomials on $\mathfrak{p}$. It is easily observed that $\mathscr{H}(\mathfrak{p})$ is a graded K-submodule of $S(\mathfrak{p})$: $\mathscr{H}(\mathfrak{p}) = \bigoplus_{k\geqslant 0} \mathscr{H}^k(\mathfrak{p})$, where $\mathscr{H}^k(\mathfrak{p}) := \mathscr{H}(\mathfrak{p}) \cap S^k(\mathfrak{p})$ is K-stable.

A result of Kostant and Rallis (cf. [12, p. 381]) says that the multiplication $(h, j) \to hj$ $(h \in \mathscr{H}(\mathfrak{p}),\ j \in S(\mathfrak{p})^K)$ gives a K-isomorphism

$$\mathscr{H}(\mathfrak{p}) \otimes S(\mathfrak{p})^K \simeq S(\mathfrak{p}), \tag{5.22}$$

where $S(\mathfrak{p})^K$ is the algebra of $\mathrm{Ad}(K)$-fixed elements of $S(\mathfrak{p})$. This implies

$$S(\mathfrak{p}) = \mathscr{H}(\mathfrak{p}) \oplus (\mathscr{H}(\mathfrak{p}) \otimes S(\mathfrak{p})^K_+) \tag{5.23}$$

as K-modules, with $S(\mathfrak{p})^K_+ = S(\mathfrak{p})^K \cap \mathfrak{p}S(\mathfrak{p})$ as before. The linear projection from $S(\mathfrak{p})$ to $\mathscr{H}(\mathfrak{p})$ along this decomposition will be denoted by α.

For any Harish-Chandra $(\mathfrak{g}, K)$-module H, we can and do take a finite-dimensional generating subspace $H_0 \subset H$ of the form

$$H_0 = \bigoplus_{\delta \in \Phi} H(\delta) \tag{5.24}$$

for a finite subset Φ of $\hat{K}$ (= the unitary dual of K), where $H(\delta)$ denotes the δ-isotypic component of H. Noting that H_0 is stable under K and $U(\mathfrak{g})^K$, one sees that the associated graded $(S(\mathfrak{g}), K)$-module

$M = \mathrm{gr}(H; H_0) = \bigoplus_k M_k$ is annihilated by $\mathfrak{k}$ and $S(\mathfrak{p})^K_+$. Hence it follows from (5.23) that $M = \mathscr{H}(\mathfrak{p}) M_0$ and that

$$\beta: \mathscr{H}(\mathfrak{p}) \otimes M_0 \ni h \otimes v \to hv \in M \tag{5.25}$$

gives a surjective K-homomorphism (cf. [18, proof of Prop. 5.5]). Note that $H_0 \simeq M_0$ as K-modules.

Now let N be any closed subgroup of G with complexified Lie algebra $\mathfrak{n}$. We set

$$\mathscr{H}(\mathfrak{p}; \mathfrak{n}) = \mathscr{H}(\mathfrak{p})/\alpha(p[\mathfrak{n}]\, S(\mathfrak{p})), \tag{5.26}$$

where $p[\mathfrak{n}]$ denotes the image of $\mathfrak{n}$ by the projection $\mathfrak{g} \to \mathfrak{p}$ along $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$ is a $(K \cap N)$-module.

We can relate the $(K \cap N)$-module $M/\mathfrak{n}M$ with $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$ as follows.

PROPOSITION 5.4. (1) *The K-homomorphism β in (5.25) naturally induces a surjective $(K \cap N)$-module map*

$$\mathscr{H}(\mathfrak{p}; \mathfrak{n}) \otimes M_0 \to M/\mathfrak{n}M. \tag{5.27}$$

(2) *If $\mathscr{N}(\mathfrak{p}) \cap \mathfrak{n}^\perp = (0)$, the space $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$ is finite-dimensional.*

Proof. Noting that $(\mathfrak{k} + S(\mathfrak{p})^K_+)M = (0)$, we get

$$\alpha(p[\mathfrak{n}]\, S(\mathfrak{p}))\, M_0 = p[\mathfrak{n}]\, S(\mathfrak{p})\, M_0 = \mathfrak{n} S(\mathfrak{p})\, M_0 = \mathfrak{n} M.$$

This proves (1).

(2) Observe that the associated variety $\mathscr{V}(L)$ (see Theorem 1.1) of graded $S(\mathfrak{p})$-module

$$L := S(\mathfrak{p})/(p[\mathfrak{n}] + S(\mathfrak{p})^K_+)\, S(\mathfrak{p})$$

equals $(S(\mathfrak{p})^K_+)^\# \cap p[\mathfrak{n}]^\perp = \mathscr{N}(\mathfrak{p}) \cap \mathfrak{n}^\perp = (0)$ (by assumption). Hence one gets $\dim L < \infty$ by Corollary 1.1. This shows that $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$ is of finite dimension, because the projection $\alpha: S(\mathfrak{p}) \to \mathscr{H}(\mathfrak{p})$ naturally gives rise to a linear isomorphism from L onto $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$. Q.E.D.

The above proposition, combined with Theorem 5.3, allows us to estimate the intertwining number $I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta))$ in (5.16) by means of $\mathscr{H}(\mathfrak{p}; \mathfrak{n})$ and $H_0 \simeq M_0$ as in

COROLLARY 5.3. *Let H be a Harish-Chandra $(\mathfrak{g}; K)$-module and let $\mathscr{A}(G; \eta)$ be the G- and $U(\mathfrak{g})$-module analytically induced from a smooth N-representation (η, E). Then one has*

$$I_{U(\mathfrak{g})}(H, \mathscr{A}(G; \eta)) \leqslant I_{K \cap xNx^{-1}}(\mathscr{H}(\mathfrak{p}; x \cdot \mathfrak{n}) \otimes H_0, E_x) \tag{5.28}$$

for each $x \in G$. Here $M = \mathrm{gr}(H; H_0)$ with H_0 in (5.24), and xNx^{-1} acts on $E_x = E$ by $xnx^{-1} \to \eta(n)$ $(n \in N)$.

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